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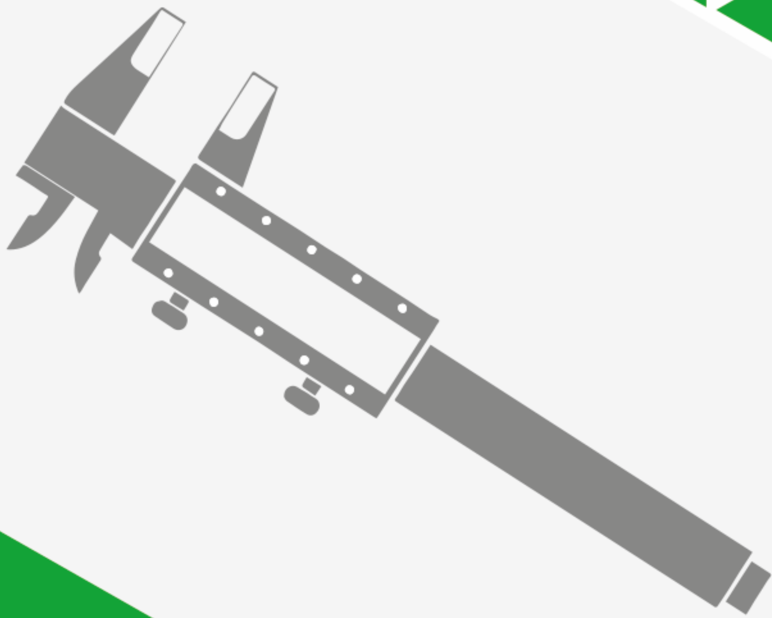

GUIDELINES ON

EVALUATION AND EXPRESSION OF MEASUREMENT UNCERTAINTY

WAQSP West Africa Quality System Programme
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ECOWAQ

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INDUSTRIAL DEVELOPMENT ORGANIZATION

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1 Introduction

More and more, it is recognised that a proper assessment of measurement uncertainty is an integral part of quality management and measurement costs. The evaluation of the measurement uncertainties allows to better understand the relative importance of different influence quantities on measurements process.

This guide provides guidance for the assessment of uncertainty contributions, the expression of the measurement result and is intended to be applicable to most measurement results of calibration laboratories and related fields.

The purpose of a measurement is to determine the value of the measurand, that is, the value of the particular quantity to be measured. The measurement result is only an approximation or estimate of the value of the measurand and is therefore complete only if it is accompanied by a statement of the uncertainty associated with that estimate.

This document gives elaborate examples illustrating the application of the method described in this guide in specific measurement problems of calibration. Evaluation of measurement uncertainty will also be addressed in other specific areas (calibration methods with specific elaborate examples).

The Best Measurement Capability (BMC) is defined as the smallest measurement uncertainty that a laboratory can achieve as part of its accreditation, when performing calibrations of measuring instrument. The evaluation of the best measurement capability of accredited calibration laboratories should be based on the method described in this document but should normally be substantiated or confirmed by experimental evidence. It will also help accreditation bodies to assess the best measurement capability and can be cited as a reference in the development of accreditation guides.

The National Metrology Institutes (MNIs) of the ECOWAS Member States will also be able to rely on this Guide, as part of the development of their Measurement and Calibration (CMC) capabilities and their publications in the KCDB, the International Bureau of Weights and Measures (BIPM) database.

2 Scope

The purpose of this document is to harmonize methods for evaluating measurement uncertainties. The evaluation method described in this document is in accordance with the Guide for the Expression of Measurement Uncertainties (GUM). The GUM establishes general rules for the evaluation and expression of measurement uncertainty that can be followed in most physical measurement areas. This document focuses on the best method of measurement for calibration laboratory and describes a harmonized method for evaluation and determination of measurement uncertainty. It covers the following topics:

- basic definitions;
- methods for evaluating uncertainty in the measurement of input quantities (Type A and Type B);
- The relationship between the measurement uncertainty of the output quantities and the measurement uncertainty of the input quantities (combined standard uncertainty);
- The expanded uncertainty of the measurement of the output quantities;
- the expression of the result of the measurement;
- Presentation of a step-by-step method to evaluate the measurement uncertainty and express the result of measurement.

3 Reference Documents

[1] (VIM) JCGM 100: 2008 GUM 1995 avec corrections mineures: Évaluation des données de mesure — Guide pour l'expression de l'incertitude de mesure

[2] Vocabulaire international de métrologie (VIM) – Concepts fondamentaux et généraux et termes associés (3ème édition), version 2008 avec corrections mineures

4 General Aspects on Measuring and Defining the Concept of Uncertainty

Measurement uncertainties arise from errors made during measurement operations. An error in the result of the measurement is considered to have two components, namely a random component and a systematic component. Random errors are likely the result of unpredictable or stochastic temporal and spatial variations in influence quantities. The systematic error comes from a recognised effect of a magnitude of influence on a measurement result.

The uncertainty of the result of a measurement generally consists of several components that can be grouped into two types according to the method used to estimate their numerical values:

- Type A - those evaluated by statistical methods,
- Type B - those evaluated by other means.

Before addressing the issue, it would be desirable to provide definitions of the terms necessary for a better understanding of the subject.

4.1 Terms and definitions

4.1.1 General metrological terms

Best measurement capability:

The smallest measurement uncertainty that a laboratory can obtain as part of its accreditation for more or less routine calibration operations.

The term "uncertainty"

The word "uncertainty" means doubt. Thus, in its broadest sense, "measurement uncertainty" means doubt about the validity of the result of a measurement. As there are no words for this general concept of uncertainty and for specific quantities that provide quantitative measures of the concept, eg standard deviation, the use of the word "uncertainty" is imposed on both different meanings.

Uncertainty (of measurement)

Parameter, associated with the result of a measurement, which characterises the dispersion of the values which could reasonably be attributed to the measurand.

NOTE 1: The parameter may be, for example, a standard deviation (or a given multiple of it), or the half-width of an interval having a stated level of confidence.

NOTE 2: Uncertainty of measurement comprises, in general, many components. Some of these components may be evaluated from the statistical distribution of the results of series of measurements and can be characterised by experimental standard deviations. The other components, which also can be characterised by standard deviations, are evaluated from assumed probability distributions based on experience or other information

NOTE 3: It is understood that the result of the measurement is the best estimate of the value of the measurand, and that all components of uncertainty, including those arising from systematic effects, such as components associated with corrections and reference standards, contribute to the dispersion.

4.1.2 Specific terms

Standard uncertainty
Uncertainty of the result of a measurement expressed as a standard deviation.

Type A evaluation (of uncertainty)
Method of evaluation of uncertainty by the statistical analysis of series of observations.

Type B evaluation (of uncertainty)
Method of evaluation of uncertainty by means other than the statistical analysis of series of observations.

Combined standard uncertainty
Standard uncertainty of the result of a measurement when that result is obtained from the values of a number of other quantities, corresponding to the positive square root of a sum of terms, the terms being the variances or covariances of these other quantities weighted according to how the measurement result varies with changes in these quantities.

Expanded uncertainty

Quantity defining an interval about the result of a measurement that may be expected to encompass a large fraction of the distribution of values that could reasonably be attributed to the measurand.

NOTE 1: The fraction may be viewed as the coverage probability or level of confidence of the interval.
NOTE 2: To associate a specific level of confidence with the interval defined by the expanded uncertainty requires explicit or implicit assumptions regarding the probability distribution characterised by the measurement result and its combined standard uncertainty. The level of confidence that may be attributed to this interval can be known only to the extent to which such assumptions may be justified.
NOTE 3: The expanding uncertainty is also known as *global uncertainty.*

Coverage factor

Numerical factor used as a multiplier of the combined standard uncertainty in order to obtain an expanded uncertainty.

NOTE: A coverage factor, k, has a value typically in the range 2 to 3.

Relative standard uncertainty of measurement

The standard uncertainty of a quantity divided by the estimate of that quantity.

Measurement

The objective of a **measurement** is to determine the **value** of the **measurand**, ie the value of the **particular quantity** to be measured. As a result, a measurement begins with an appropriate definition of the measurand, the **measurement method** and the **measurement procedure**.

NOTE: The terms "value of a measurand" (or of a quantity) and "true value of a measurand" (or of a quantity) are considered to be two equivalent terms.

In general, the **result of a measurement** is only an approximation or **estimate** of the value of the measurand and thus is complete only when accompanied by a statement of the **uncertainty** of that estimate.

In practice, the required specification or definition of the measurand is dictated by the required **accuracy of measurement**. The measurand should be defined with sufficient completeness with respect to the required accuracy so that for all practical purposes associated with the measurement its value is unique. It is in this sense that the expression “value of the measurand” is used in this Guide.

If the length of a nominally one-meter long steel bar is to be determined to micrometre accuracy, its specification should include the temperature and pressure at which the length is defined. Thus the measurand should be specified as, for example, the length of the bar at 25.00 °C* and 101 325 Pa (and, in addition with any other defining parameters deemed necessary, such as the way the bar is to be supported). However, if the length is to be determined to only millimetre accuracy, its specification would not require a defining temperature or pressure or a value for any other defining parameter.

NOTE: Incomplete definition of the measurand can give rise to a component of uncertainty sufficiently large that it must be included in the evaluation of the uncertainty of the measurement result.

Errors, effects and corrections

Measurement error
Difference between the measured quantity value minus a reference quantity value.

In general, a measurement has imperfections that give rise to an **error** in the measurement result. Traditionally, an error is viewed as having two components, namely, a **random** component and a **systematic** component.

NOTE: The concept of error is ideal and errors cannot be known exactly.

Random error

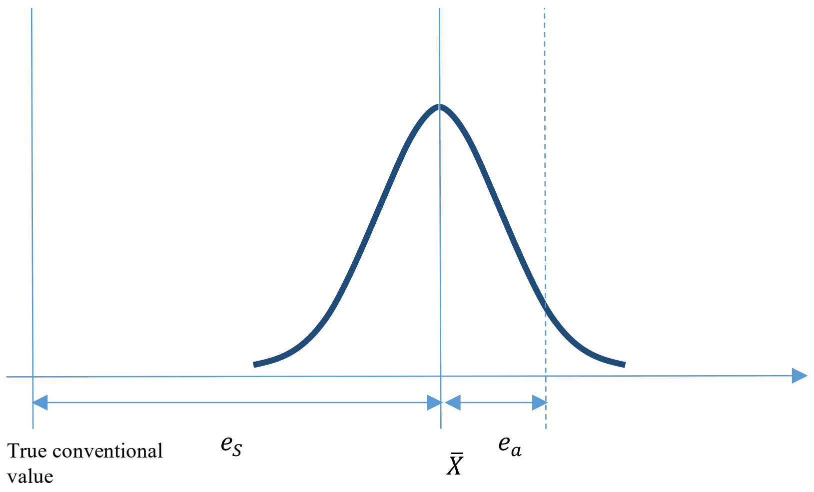
Random error presumably arises from unpredictable or stochastic temporal and spatial variations of influence quantities. The effects of such variations, hereafter termed *random effects*, give rise to variations in repeated observations of the measurand. Although it is not possible to compensate for the random error of a measurement result, it can usually be reduced by increasing the number of observations. Its **expectation** or **expected value** is equal to zero.

NOTE: The experimental standard deviation of the arithmetic mean or average of a series of observations *is not* the random error of the mean, although it is designated as such in some publications. It is instead a measure of the *uncertainty* of the mean due to random effects. The exact value of the error in the mean arising from these effects cannot be known.

Systematic error

Systematic error, like random error, cannot be eliminated but it too can often be reduced. If a systematic error arises from a recognised effect of an influence quantity on a measurement result, hereafter termed a *systematic effect*, the effect can be quantified and, if it is significant in size relative to the required accuracy of the measurement, a **correction** or **correction factor** can be applied to compensate for the effect. It is assumed that, after correction, the expectation or expected value of the error arising from a systematic effect is zero.

It is assumed that the result of a measurement has been corrected for all recognised significant systematic effects and that every effort has been made to identify such effects.



Measurement and calibration concept

Calibration

Operation that, under specified conditions, in a first step, establishes a relation between the **quantity values** with **measurement uncertainties** provided by measurement standards and corresponding indications with associated measurement uncertainties and, in a second step, uses this information to establish a relation for obtaining a **measurement result** from an indication.

4.1.3 Instrumental bias

Instrumental bias is the difference of value read minus a reference quantity value.

E_a = Reading - reference quantity value

Reading is the value read on the measuring instrument.
Reference quantity value is the value of the standard that was used to make the measurement.

4.1.4 Correction

The result of a measurement when using a calibrate or verified instrument is:

$$\boldsymbol{Corrected\_reading = Reading - E_a}$$

$$\boldsymbol{C = -E_a}$$

$$\boldsymbol{Corrected\_reading = Reading + C}$$

5 The GUM method

5.1 Sources of uncertainty (The 5M)

There are in practice many possible sources of uncertainty in a measurement, including:

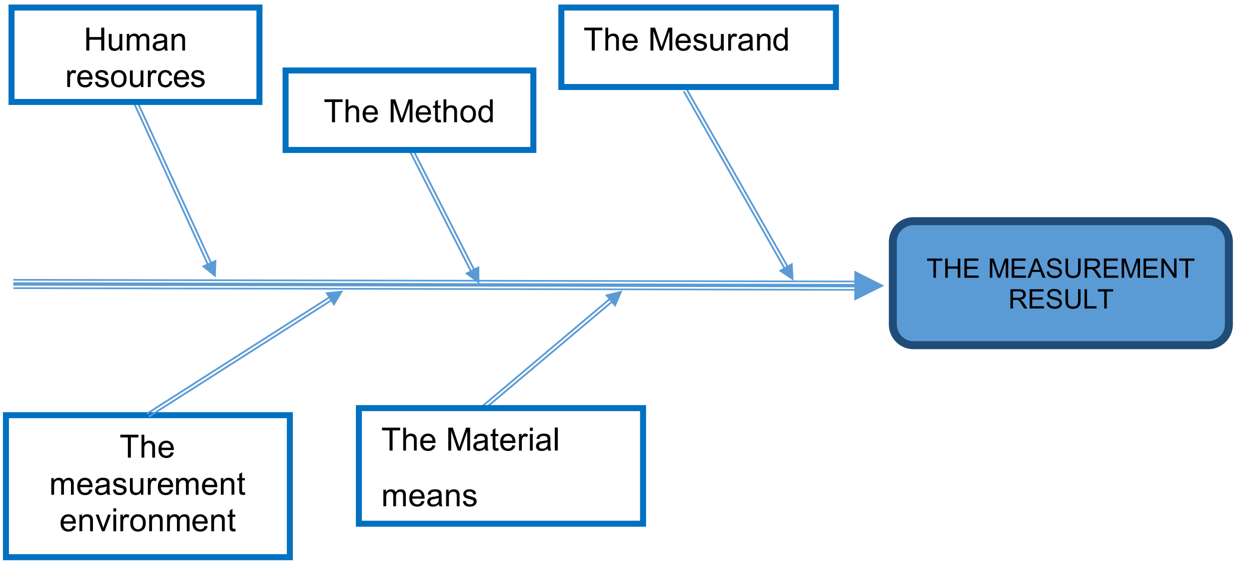
a) Incomplete definition of the measurand;
b) Imperfect realization of the definition of the measurand;
c) Unrepresentative sampling - the measured sample may not represent the defined measurand;
d) Insufficient knowledge of the effects of environmental conditions on the imperfect measurement or measurement of environmental conditions;
e) Bias due to observer for reading analogue instruments;
f) finite instrument resolution or mobility threshold;
g) Inaccurate values of reference standards and reference materials;
h) Inaccurate values of constants and other parameters obtained from external sources and used in the data processing algorithm;
i) Approximations and assumptions introduced in the measurement method and procedure;
j) Variations between repeated observations of the measurand in the same conditions.

5.1.1 Characterisation of the measurement process

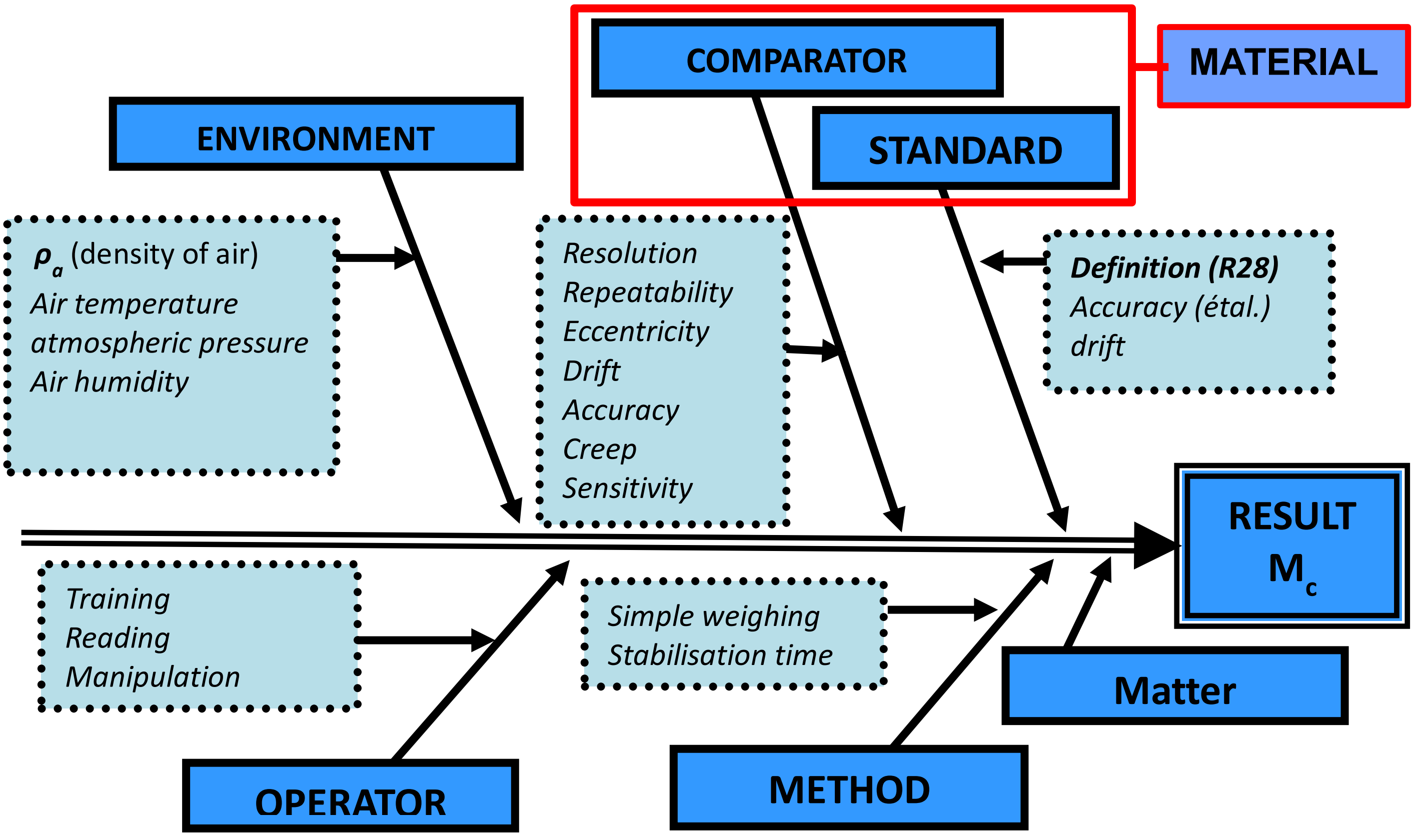
The 5M method includes:

- The Mesurand (Matter)
- The Method (operating mode)
- The material means of measurement
- Human resources (Labour Force)
- The environment of the measurement

Identification of the actual or potential causes of variation in the measurement result:
Cause and Effect Diagram (or Ishikawa Diagram or Fishbone Diagram)



Practical case of calibrating of a balance



Characterise the measurement process:

- Describe the procedure chosen,
- Establish a model of measurement;

- identify the causes of error;
- Calculate the value of the error component associated with each cause,
- Make the decision on whether or not to apply the corresponding correction (if one needs to correct, who corrects? How?)

5.2 Type A Evaluation and Type B Evaluation

The uncertainties are classified into two categories based on their evaluation method, "A" and "B". These categories apply to *uncertainty* and are not substitutes for the words "random" and "systematic". The uncertainty of a correction for a known systematic effect can be obtained in some cases by a Type A assessment and, in other cases, by a Type B assessment; it may be the same for the uncertainty that characterises a random effect.

5.2.1 Type A Standard Uncertainty

The estimated variance u^2 that characterises a component of the uncertainty obtained from a Type A evaluation is calculated from series of repeated observations and is the familiar statistically estimated variance. The **estimated standard** deviation u, the square root of u^2, is thus $u = s$ and, for convenience, is sometimes called the *Type A standard uncertainty*.

5.2.2 Type B Standard Uncertainty

For a component of the uncertainty obtained from a Type B assessment, the estimated variance u^2 is evaluated using available knowledge and the estimated standard deviation u is sometimes referred to as the *Type B standard uncertainty*.

Thus a Type B standard uncertainty from a **probability density function** (or simply a **probability density**) deduced from a **distribution of the population** (or frequency distribution) observed while obtaining an uncertainty type B type from an assumed probability density, based on the degree of belief that an event occurs [often called subjective **probability**]. Both approaches use classical interpretations of probability.

NOTE: A Type B assessment of a uncertainty component is usually based on a relatively reliable set of information.

5.2.3 Combined standard uncertainty

The standard uncertainty of the result of a measurement, when that result is obtained from the values of a number of other quantities, is termed *combined standard uncertainty* and denoted by u_c. It is the estimated standard deviation associated with the result and is equal to the positive square root of the combined variance obtained from all variance and **covariance** components, however evaluated, using what is termed in this Guide the *law of propagation of uncertainty.*

5.2.4 Coverage factor k

To meet the needs of some industrial and commercial applications, as well as requirements in the areas of health and safety, an *expanded uncertainty U* is obtained by multiplying the combined standard uncertainty *uc* by a *coverage factor k*. The intended purpose of *U* is to provide an interval about the result of a measurement that may be expected to encompass a large fraction of the distribution of values that could reasonably be attributed to the measurand. The choice of the factor *k*, which is usually in the range 2 to 3, is based on the coverage probability or level of confidence required of the interval.

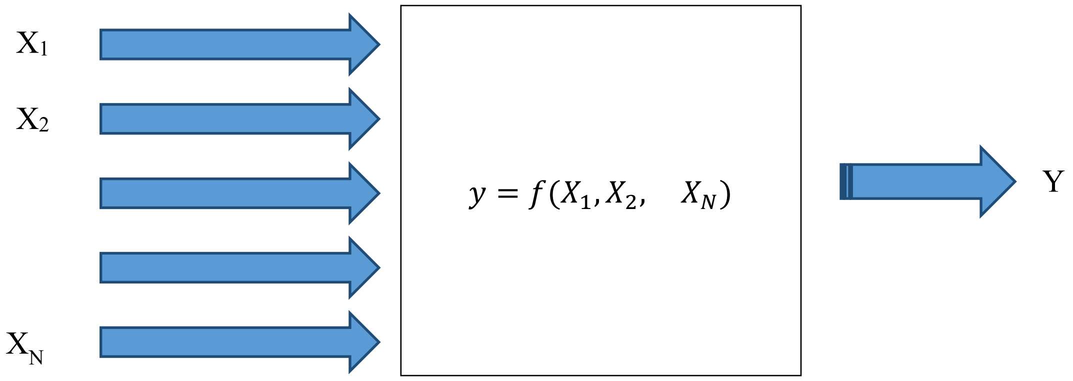
NOTE: The coverage factor *k* is always to be stated, so that the standard uncertainty of the measured quantity can be recovered for use in calculating the combined standard uncertainty of other measurement results that may depend on that quantity.

5.3 Practical considerations

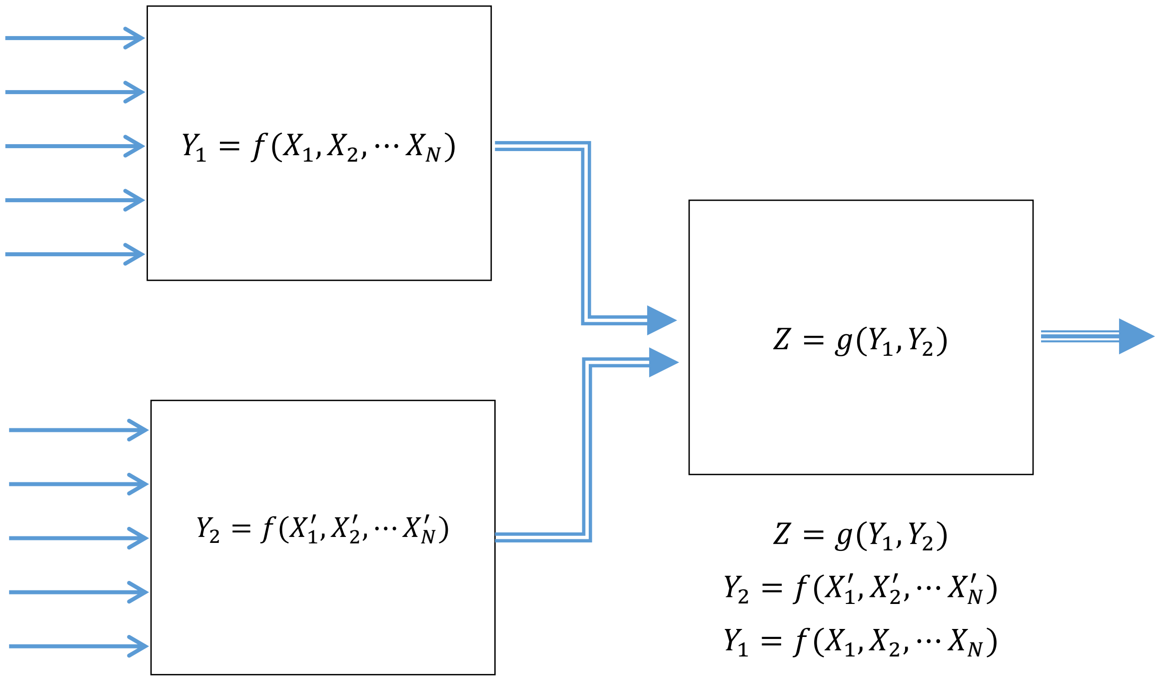
5.3.1 Mathematical model of measurement

In many cases, a measurand Y is not measured directly but it is determined from N other quantities $X_1, X_2, ..., X_N$ through a functional relation f:

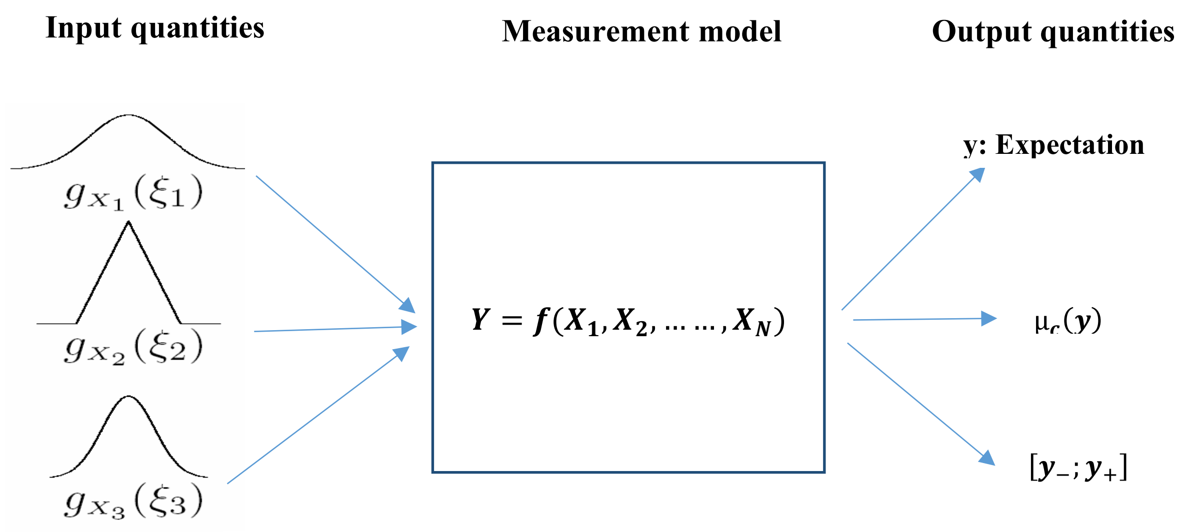
$$\boldsymbol{Y = f(X_1, X_2, \dots\dots, X_N)}$$



Single-process model



Model with intermediate processes



<u>Mathematical model of the measurement process</u>

The uncertainty of a measurement result is usually evaluated by using a mathematical model of the measurement and the law of propagation of uncertainty.

In a series of observations, the k^{th} observed value of X_i is denoted $X_{i,k}$; thus, if a resistance is denoted R, the k^{th} observed value of the resistance is denoted R_k.

The estimate of *Xi* (of his mathematical expectation) is noted *xi.*

If a potential difference V is applied to the terminals of a temperature-dependent resistor that has a resistance R_0 at the defined temperature t_0 and a linear temperature coefficient of resistance α, the power P (the measurand) dissipated by the resistor at the temperature t depends on V, R_0, α, and t according to the following formula:

$$P = \frac{V^2}{R_0}[1 + \alpha(t - t_0)]$$

Mathematical model	Mathematical formula
$P = f(V, R_0, \alpha, t)$	$P = \frac{V^2}{R_0}[1 + \alpha(t - t_0)]$
$P = f(n, R, T, V)$	$P = \frac{nRT}{V}$
$L = f(L_0, \alpha, T)$	$L = L_0 + \alpha(T - T_0)$

The *input quantities* X_1, X_2, ..., X_N upon which the *output quantity* Y depends may themselves be viewed as measurands and may themselves depend on other quantities, including corrections and correction factors for systematic effects, thereby leading to a complicated functional relationship.

An estimate of the measurand Y, denoted by *y*, is obtained from the Equation using the input estimates x_1, x_2, ..., x_N for the values of N quantities X_1, X_2, ..., X_N. Thus, the *output estimate* y, which is the result of the measurement, is given by:

$$y = f(x_1; x_2; \ldots \ldots; x_N)$$

The estimated standard deviation associated with the output estimate or the measurement result y, called the combined standard uncertainty and noted u_c *(y)*, is determined from the estimated standard deviation associated with each input estimate. xi, called *standard uncertainty* and noted u (x_i)

Each input estimate xi as well as its associated standard uncertainty u(xi) are obtained from a distribution of possible values of the input quantity X_i. This probability distribution may be frequency based, that is, based on a series of observations $X_{i,k}$ of X_i, or it may be an *a priori* distribution.

Type A evaluations of standard uncertainty components are founded on frequency distributions while Type B evaluations are founded on *a priori* distributions.

5.3.2 Consideration of measurement uncertainties

In some cases, it is not necessary to include the uncertainty of a correction for a systematic effect in the evaluation of the uncertainty of a measurement result. Although uncertainty has been evaluated, it can be ignored if its contribution to the standard uncertainty composed of the measurement result is insignificant.

If the value of the correction itself is insignificant in relation to the combined standard uncertainty, it can also be ignored.

5.4 Evaluation of standard uncertainty

5.4.1 Basic statistical terms and concepts

Probability

Real number in the range of 0 to 1, associated with a random event

$$\boldsymbol{F(x) = Pr(X \leq x)}$$

Probability density function (for a continuous random variable)
The derivative (when it exists) of the distribution function:

$$\boldsymbol{f(x) = \frac{dF(x)}{dx}}$$

$\boldsymbol{f(x)dx}$ is “probability element”.

Correlation

The relationship between two or several random variables within a distribution of two or more random variables.

5.4.2 Elaboration of terms and concepts

Expectation

The expectation of a function g(z) over a probability density function p(z) of the random variable z is defined by

$$E[g(z)] = \int g(z)\, p(x) dz$$

Where, from the definition of p(z), ∫p(z) dz = 1. The expectation of the random variable z, denoted by μ_z, and which is also termed the expected value or the mean of z, is given by :

$$\mu_z \equiv E(z) = \int zp(z) dz$$

It is estimated statistically by z̄, the arithmetic mean or average of *n* independent observations *zi* of the random variable *z*, the probability density function of which is *p(z):*

$$\bar{z} = \frac{1}{n}\sum_{i=1}^{n} z_i$$

Variance

The variance of a random variable is the expectation of its quadratic deviation about its expectation. Thus the variance of random variable *z* with probability density function *p(z)* is given by:

$$\sigma^2(z) = \int (z - \mu_z)^2 p(z) dz$$

Where μ_z is the expectation of *z*. The variance $\sigma^2(z)$ may be estimated by

$$\delta^2(z_i) = \frac{1}{n-1}\sum_{j=1}^{n} \left(z_j - \bar{z}\right)^2$$

$$\bar{z} = \frac{1}{n}\sum_{i=1}^{n} z_i$$

and the *zi* are *n* independent observations of *z*.

The factor *n* − *1* in the expression for $\delta^2(z_i)$ arises from the correlation between z_i and z and reflects the fact that there are only *n* − *1* independent items in the set $(z_i - \bar{z})$.

If the expectation $\mu_z$of z is known, the variance may be estimated by

$$\delta^2(z_i) = \frac{1}{n-1}\sum_{i=1}^{n} (z_i - \mu_z)^2$$

The variance of the arithmetic mean or average of the observations, rather than the variance of the individual observations, is the proper measure of the uncertainty of a measurement result.

The variance of a variable z should be carefully distinguished from the variance of the mean $\bar{z}$. The variance of the arithmetic mean of a series of *n* independent observations *zi* of *z* is given by σ $\sigma^2(\bar{z}) = \sigma^2(z_i)/n$ and is estimated by the experimental variance of the mean

$$\delta^2(\bar{z}) = \frac{\delta^2(z_i)}{n} = \frac{1}{n(n-1)}\sum_{i=1}^{n}(z_i - \bar{z})^2$$

Standard deviation

The standard deviation is the positive square root of the variance. A Type A standard uncertainty is obtained by taking the square root of the statistically evaluated variance, it is often more convenient when determining a Type B standard uncertainty to evaluate a nonstatistical equivalent standard deviation first and then to obtain the equivalent variance by squaring the standard deviation.

Covariance

The covariance of two random variables is a measure of their mutual dependence. The covariance of random variables y and z is defined by:

$$cov(y,z) = cov(z,y) = E\big[[y - E(y)][z - E(z)]\big]$$

$$cov(y,z) = cov(z,y) = \iint (y - \mu_y)\,(z - \mu_z)p(y,z)dydz$$

$$= \iint (y - \mu_y)\,(z - \mu_z)p(y,z)dydz$$

Correlation coefficient

The correlation coefficient is a measure of the relative mutual dependence of two variables, equal to the ratio of their covariance to the positive square root of the product of their variances:

$$\rho(y,z) = \rho(z,y)$$

Independence

Two random variables are statistically independent if their joint probability distribution is the product of their individual probability distributions.

NOTE: If two random variables are independent, their covariance and correlation coefficient are zero, but the converse is not necessarily true.

5.4.3 Type A evaluation of standard uncertainty

The best available estimate of the expectation or expected value μ_q of a quantity q that varies **randomly**, and for which *n* independent observations have been obtained under the same conditions of measurement, is the **arithmetic mean or average** $\bar{q}$ of the n observations:

$$\bar{q} = \frac{1}{n}\sum_{k=1}^{n} q_k$$

The experimental variance of the observartions, which estimate the variance $\sigma^2$of the probability law of *q*, is given by:

$$\delta^2(q_k) = \frac{1}{n-1}\sum_{j=1}^{n}\left(q_j - \bar{q}\right)^2$$

This estimate of variance and its positive square root $\delta(q_k)$, termed the **experimental standard deviation**, characterise the variability of the observed values q_k, or more specifically, their dispersion about their mean $\bar{q}$.

The best estimate of $\sigma^2(\bar{q}) = \sigma^2/n$, the variance of the mean, is given by:

$$\delta^2(\bar{q}) = \frac{\delta^2(q_k)}{n}$$

The experimental variance of the mean $\delta^2(\bar{q})$ and **the experimental standard deviation of the mean$\boldsymbol{\delta(\bar{q})}$**, is equal to the positive square root of $\delta^2(\bar{q})$, quantify how well $\bar{q}$ estimates the expectation μ_q of *q*, and either may be used as a measure of the uncertainty of*q*.

Thus, for an input quantity X_i determined from n independent repeated observations$X_{i,k}$, the standard uncertainty $u(x_i)$ of its estimate $x_i = \bar{X}_i$ is $u(x_i) = \delta(\bar{X}_i)$, with $\delta^2(\bar{X}_i)$ calculated according to the above Equation.

And, $u^2(x_i) = \delta^2(\bar{X}_i)$ and $u(x_i) = \delta(\bar{X}_i)$ are sometimes called a *Type A variance and a Type A standard uncertainty*, respectively.

NOTE 1: The number of observations n must be large enough to ensure that $\bar{q}$ provides a reliable estimate of the expectation μ_q.

The degrees of freedom v_i of $u(x_i)$, equal to **n − 1** in the simple case where $x_i = \bar{X}_i$ and $u(x_i) = s(\bar{X}_i)$ are calculated from *n* independent observations and, should always be given when Type A evaluations of uncertainty components are given.

5.4.4 Type B evaluation of standard uncertainty

For an estimate x_i of an input quantity X_i that has not been obtained from **repeated observations**, the associated estimated variance $u^2(x_i)$ or the standard uncertainty $u(x_i)$ is evaluated by **scientific judgement** based on all of the **available information** on the **possible variability** of X_i . The pool of information may include:

- previous measurement data;
- experience with or general knowledge of the behaviour and properties of relevant materials and instruments;
- manufacturer's specifications;
- data provided in calibration and other certificates;
- uncertainties assigned to reference data taken from handbooks.

For convenience, $u^2(x_i)$ and $u(x_i)$ evaluated in this way are sometimes called a ***Type B variance*** and a ***Type B standard uncertainty***, respectively.

A Type B assessment of standard uncertainty uses experience and general knowledge, and it is a skill that is learned through practice.

If the estimate x_i is taken from a **manufacturer's specification, calibration certificate, handbook**, or other source and its quoted uncertainty is stated to be a **particular multiple** of a standard deviation, the standard uncertainty $u(x_i)$ is simply the **quoted value divided by the multiplier**, and the estimated variance $u^2(x_i)$ is the square of that quotient.

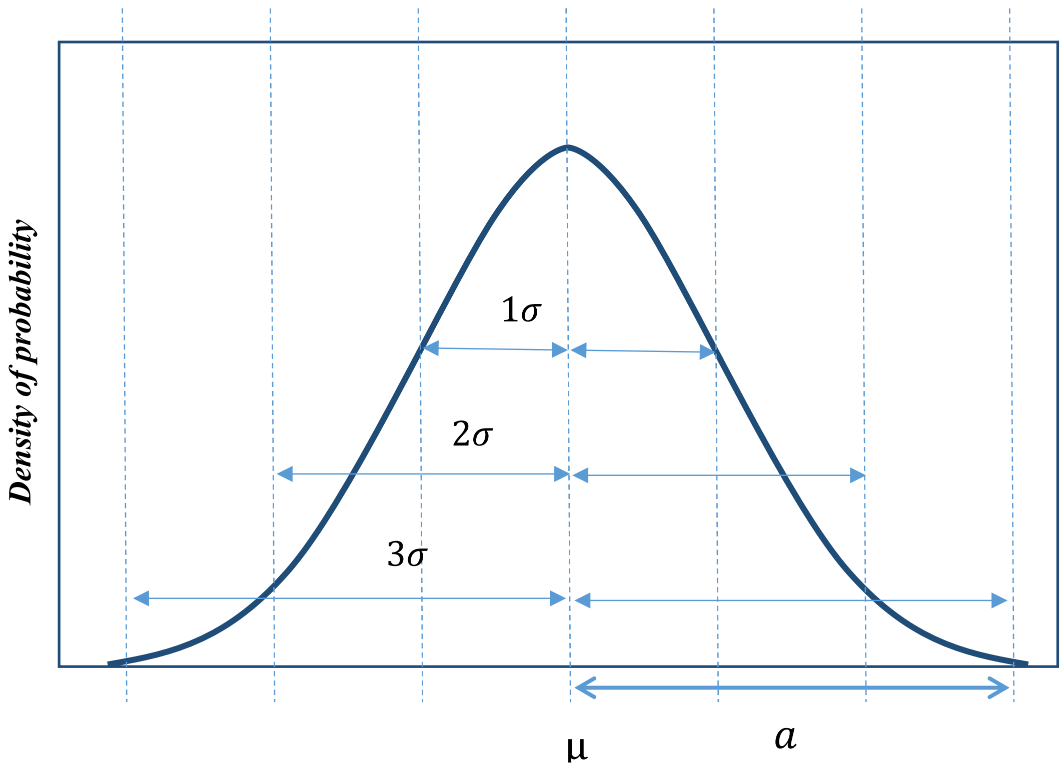
It can be assumed that a **normal law** has been used.

5.4.5 Distribution functions

5.4.5.1 Normal distribution

The normal law is a law of probability of a continuous random variable X, whose probability density is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$$

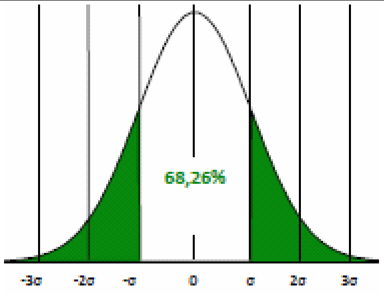
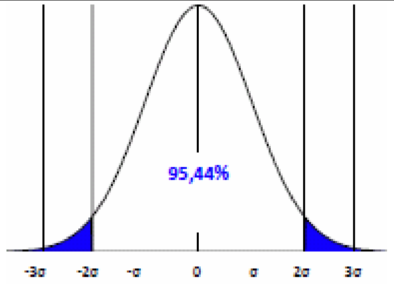
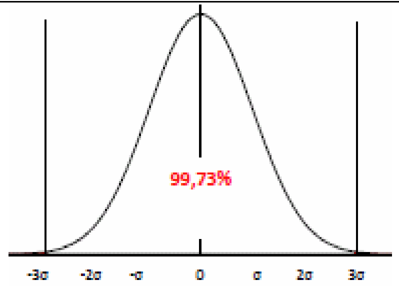


$\mu \pm 1\sigma$ corresponds to 68.27 % of observations

$\mu \pm 2\sigma$ corresponds to 95.45 % of observations

$\mu \pm 3\sigma$ corresponds to 99.73% of observations

$a = 3\sigma$

		
$(-\sigma \leq X - m \leq \sigma)$ Probability ≈ 0.68	$(-2\sigma \leq X - m \leq 2\sigma)$ Probability ≈ 0.95	$(-3\sigma \leq X - m \leq 3\sigma)$ Probability ≈ 0.997

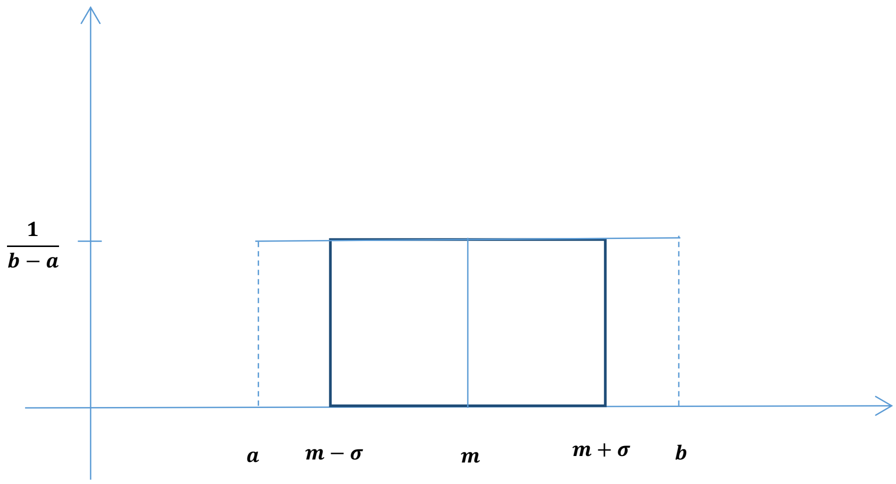
For a **normal expectation** of mathematical expectation μ and standard deviation σ, the **interval μ ± 3σ** covers approximately **99.73** percent of the possible values of the law. If the upper and lower limits a + and a- then set limits at 99.73 percent instead of 100 percent, then

$u^2(x_i) = a^2/9$ and $u(x_i) = a/3$

5.4.5.2 Rectangular distribution (or Uniform distribution)

A variable X follows a uniform distribution over an interval [a, b] if its density function f is defined by:

$$f(x) = \frac{1}{b-a} \; for \; a \leq x \leq b \;\; and \; f(x) = 0 \;\; if \; not$$



It is easy to show that:
The variance:

$$V(X) = \frac{(b-a)^2}{12}$$

The standard deviation: $\sigma = \frac{(b-a)}{2\sqrt{3}}$

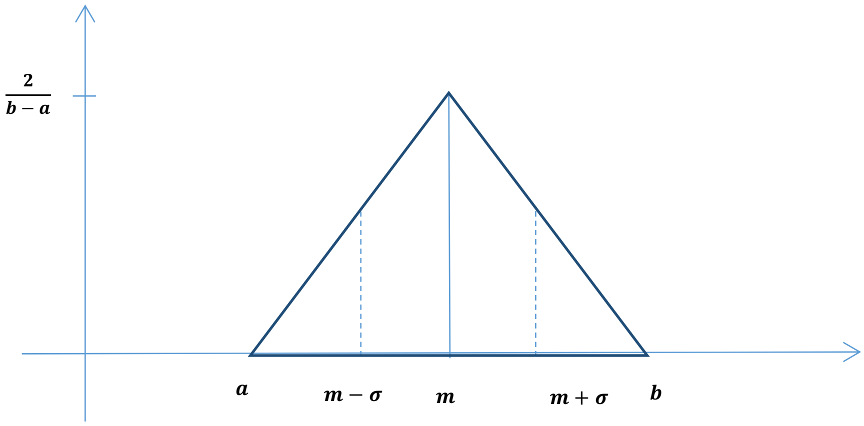
5.4.5.3 Triangular distribution

A variable X follows a triangular distribution over an interval [a, b] if its density function f is defined by:

$$f(x) = \frac{4(x-a)}{(b-a)^2} \; for \; a \leq x \leq \frac{b+a}{2}$$

$$f(x) = \frac{4(b-x)}{(b-a)^2} \; for \; \frac{b+a}{2} \leq x \leq b$$

$$f(x) = 0 \; if \; not$$

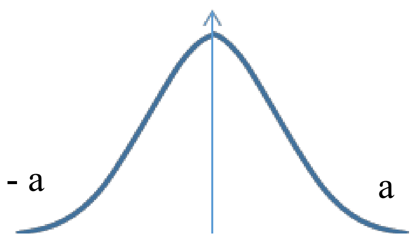
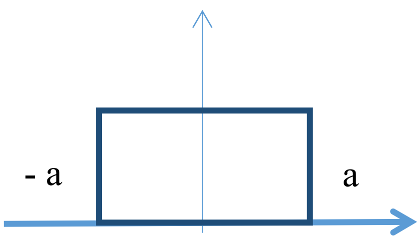


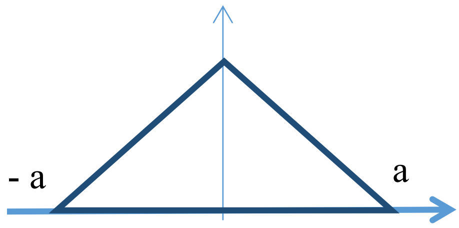
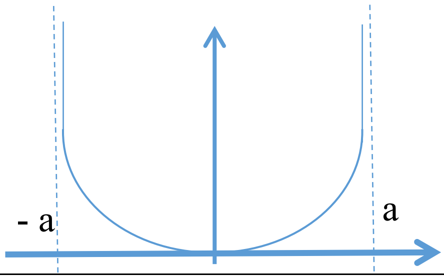
We show that the expectation or the mean of X = $\frac{b+a}{2}$

And the variance $V(X) = \frac{(b-a)^2}{24}$

The standard deviation $\sigma = \frac{(b-a)}{2\sqrt{6}}$

EXAMPLES OF DISTRIBUTION FUNCTIONS FOR TYPE B METHOD

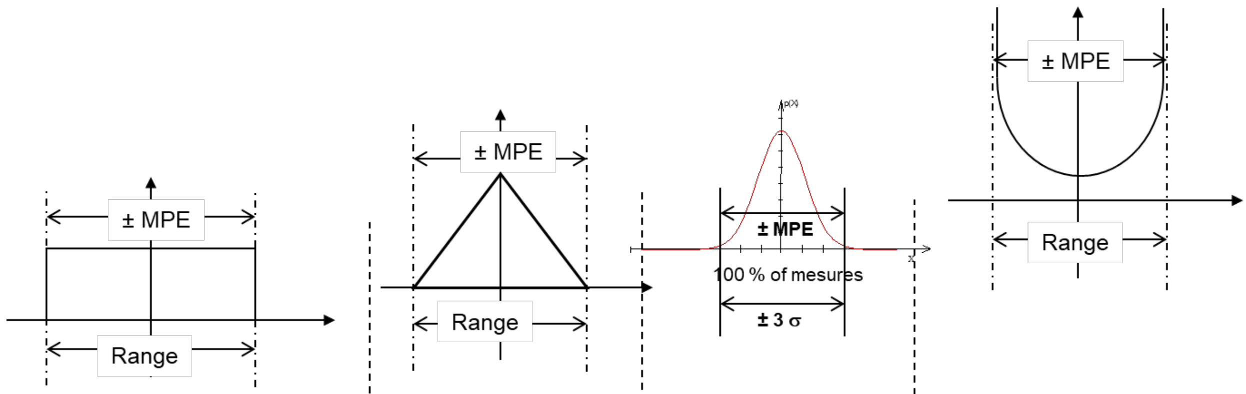
Law	Distribution	Variance	Standard deviation
NORMALE		$\frac{d^2}{36} = \frac{a^2}{9}$	$\frac{d}{6} = \frac{a}{3}$
RECTANGULAR OR UNIFORM		$\frac{d^2}{12} = \frac{a^2}{3}$	$\frac{d}{\sqrt{12}} = \frac{a}{\sqrt{3}}$

TRIANGULAR		$\frac{d^2}{24} = \frac{a^2}{6}$	$\frac{d}{\sqrt{24}} = \frac{a}{\sqrt{6}}$
U-SHAPED (ARCSINE)		$\frac{d^2}{8} = \frac{a^2}{2}$	$\frac{d}{\sqrt{8}} = \frac{a}{\sqrt{2}}$

Probability distribution function most commonly used in the evaluation of the components of type B uncertainties areas as follows:

- The normal distribution,
- Uniform or rectangular distribution
- The Triangular distribution
- The U-shaped (arcsine) distribution

SUMMARY OF METHODS TYPE B
(EMT and range)



Rectangular	Triangular	Normal	Arcsine
$u_B = \frac{Range}{2\sqrt{3}}$	$u_B = \frac{Range}{2\sqrt{6}}$	$u_B = \frac{Range}{6}$	$u_B = \frac{Range}{2\sqrt{2}}$
$u_B = \frac{MPE}{\sqrt{3}}$	$u_B = \frac{MPE}{\sqrt{6}}$	$u_B = \frac{MPE}{3}$	$u_B = \frac{MPE}{\sqrt{2}}$

A FEW CLASSICAL CASES OF THE TYPE B METHOD

N°	Particular case	Standard uncertainty
1	A measuring instrument having a calibration certificate giving an uncertainty U is used as a multiple of the standard uncertainty: $$U = k.u$$ k is generally known as the " coverage factor", where U is the "expanded uncertainty". The operator can calculate the standard uncertainty: $$u = \frac{U}{k}$$ when the coverage factor is given (for example, for calibration certificates issued by a calibration lab accredited by an accreditation body member of ECORAS, the coverage factor is conventionally set at K = 2)	$u = \frac{U}{k}$
2	A measuring instrument having a verification certificate attesting compliance with specifications of (manufacturer, regulatory, normative or contractual) defining an interval $[x - t\ , x + t]$ of width equal to twice a maximum permissible error t symmetrically distributed is used. around reading x The corresponding standard uncertainty is obtained by not privileging any value belonging to this interval, thus associating a rectangular distribution, in which case $$u = \frac{t}{\sqrt{3}}$$	$u = \frac{t}{\sqrt{3}}$
3	The uncertainty due to the error of non-linearity or hysteresis is treated as above insofar as the value of the error (thus the correction to be made) is not known at the point of operation. Since the manufacturer defines a "maximum error of nonlinearity" *nl* or a "maximum error of hysteresis" h, the corresponding uncertainty components will be given respectively by: $$u = \frac{nl}{\sqrt{12}}$$ $$u = \frac{h}{\sqrt{12}}$$	Uncertainty of non-linearity $u = \frac{nl}{\sqrt{12}}$ Hysteresis uncertainty $u = \frac{h}{\sqrt{12}}$
4	For a system having a finite resolution (for example a digital instrument with a resolution q), a rectangular distribution is associated so that the standard uncertainty resulting from this resolution is :	$u = \frac{q}{\sqrt{12}}$

	$u = \frac{q}{\sqrt{12}}$	
5	The measurement of the temperature in a room with a temperature variation of ΔT: We associate a U-shaped (arcsine) distribution: $u = \frac{\Delta T}{2\sqrt{2}}$	$u = \frac{\Delta T}{2\sqrt{2}}$
6	In the field of advanced manufacturing a triangular distribution can be use	$u = \frac{a}{\sqrt{6}}$
7	Limited precision calculations The rounding or truncation of numbers that occurs in automatic data reductions by computers can also be a source of uncertainty. Consider for example a computer with a length of 16-bytes words. If, during the calculation, a number corresponding to this word length is subtracted from another number from which it differs only by the 16 th byte, there remains only one significant byte. Such cases may occur in the evaluation of "poorly conditioned" algorithms and may be difficult to predict; if the variation is δx $u = \frac{\delta x}{\sqrt{12}}$	$u = \frac{\delta x}{\sqrt{12}}$

Practical examples of application of Type B methods.

Rectangular probability distribution

The measurement accuracy of a voltmeter is ± 0.05%. The half-interval limit is 0.05% and the standard uncertainty is given by:

$$u(V) = \frac{0.05\ \%}{\sqrt{3}}$$

The resolution of the digital display of the voltage is 1 mV. Thus, the interval is 1mV and the half-interval limit is half of 1mV. The standard uncertainty is then given by:

$$u(V) = \frac{1\text{mV}}{2\sqrt{3}}$$

The hysteresis effect of an instrument is 0.1%, meaning the equivalent interval to the difference between the maximum and the minimum of the same input. The limit of the half-interval limit is equal to half of 0.1%. The standard uncertainty is given by:

$$u(R) = \frac{0.1\%}{2\sqrt{3}}$$

The maximum drift of the value of a calibration standard between calibration intervals is 0.001 pF. The history of the capacity standard of recent years shows that the value of the capacity changes by as much as 0.001 pF. The standard uncertainty is then given by:

$$u(drift) = \frac{0.001\ \text{pF}}{\sqrt{3}}$$

Probability distribution in U-shape or arcsine

The output power of a signal generator is measured by a wattmeter. The magnitude of the reflection coefficients of the signal generator and the wattmeter are respectively 0.2 and 0.091. The standard uncertainty due to disparity is given by:

$$u(m) = \frac{2\text{x}0.2\text{x}0.091}{\sqrt{2}}$$

Normal or Gaussian Probability Distribution

A calibration report indicates that the uncertainty is ± 0.1 dB with a coverage factor of 2.63. The standard uncertainty is given by:

$$u(x) = \frac{0.1\ \text{dB}}{2.63}$$

A calibration certificate indicates that the mass m_S of a stainless-steel mass standard with a nominal value of one kilogram is **1,000,000,325 g** and that "the uncertainty on this value is equal to **240 µg** at the level of **3 deviations** ".

The standard uncertainty of the mass standard is then simply:

$u(m_S) = \frac{240\ \mu\text{g}}{3}$= 80 µg

This corresponds to a relative standard uncertainty $u(m_S)/m_S$ equal to 80 10^{-9}.
The estimated variance is $u^2(m_S)$= (80 µg)^2 = 6.4 10^{-9}g^2.

The GUM recommends that the enlargement factor used should always be given in the documents cited above.

A calibration certificate indicates that the R_S value of a nominal resistance equal to ten ohms is 10.000 742 Ω ± 129 µΩ at 23 ° C and that the "indicated uncertainty of 129 µΩ defines an interval at 99 percent confidence."

The standard uncertainty on the value of the resistance can be taken as:

$u(R_S)$ = (129 µΩ)/2.58 = 50 µΩ

which corresponds to a relative standard uncertainty $u(R_S)/R_S$ de 5.0 10^{-6}. The estimated variance is $u^2(m_S)$ = (50 µΩ)^2 = 2.5 10^{-9}Ω^2.

In summary: The rectangular distribution is a reasonable default model in the absence of any other information. But, if we know that the value of the quantity in question is close to the centre of the limits, a triangular or normal distribution may be a better model.

5.5 Determination of the combined standard uncertainty

5.5.1 When input quantities are uncorrelated (input quantities are independent)

The standard uncertainty of y, where y is the estimate of the measurand Y and thus the result of the measurement, is obtained by appropriately combining the standard uncertainties of the input estimates x_1, x_2, ..., x_N

This *combined standard uncertainty* of the estimate y is denoted by $u_C(y)$

5.5.1.1 Definition of the sensitivity coefficient

For a linear function f dependent on a single variable given by $Y = f(X)$, the sensitivity coefficient is defined by

$$C_i = \frac{\partial Y}{\partial X} = \frac{\Delta Y}{\Delta X}$$

5.5.1.2 Taylor's formula

The generalized Taylor formula including nonlinear functions is as follows:

$$f(X) = f(X_0) + \left.\frac{\partial f}{\partial X}\right|_{X_0} \cdot \delta X + \frac{1}{2} \cdot \left.\frac{\partial^2 f}{\partial X^2}\right|_{X_0} + \cdots$$

For linear functions, the combined standard uncertainty $u_C(y)$ is the square root of the composition of variance $u_C^2(y)$ that is given by **the law of propagation of uncertainties:**

$$u_C^2(y) = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial x_i}\right)^2 u^2(x_i)$$

f is defined by the equation: $\boldsymbol{Y = f(X_1, X_2, \ldots\ldots, X_N)}$.
$u(x_i)$ are standard uncertainties evaluated by **Type A or Type B assessments.**

Another expression of $u_C^2(y)$ is:

$$u_C^2(y) = \sum_{i=1}^{N} [c_i u(x_i)]^2 \equiv \sum_{i=1}^{N} u_i^2(y)$$

The sensitivity coefficient is:

$$c_i = \frac{\partial f}{\partial x_i}$$

$$u_i(y) \equiv |c_i| u(x_i)$$

The combined standard uncertainty $u_C(y)$ is an **estimated standard deviation** and characterises the dispersion of values that could reasonably be attributed to the measurand Y.

When the non-linearity of f becomes significant, higher order terms must be included in the Taylor serial development:

$$u_C^2(y) = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial x_i}\right)^2 u^2(x_i) + \sum_{i=1}^{N}\sum_{j=1}^{N}\left[\frac{1}{2}\left(\frac{\partial^2 f}{\partial x_i x_j}\right)^2 + \frac{\partial f}{\partial x_i}\frac{\partial^3 f}{\partial x_i \partial x_j^2}\right] u^2(x_i) u^2(x_j)$$

If Y is of the form $Y = cX_1^{p1}X_2^{p2}\cdots X_N^{pN}$ and if the exponents *pi* are known numbers, positive or negative, the combined variance, Equation for the uncorrelated input data can be expressed in the form:

$$\left[\frac{u_c(y)}{y}\right]^2 = \sum_{i=1}^{N}\left[p_i \frac{u(x_i)}{x_i}\right]^2$$

5.6 Table of uncertainties budget:

Mathematical model			$y = f(x_1; x_2; \ldots.. x_N)$
Standard uncertainty of measurement	$u(x_i)$	standard uncertainty associated with the input value xi xi	
	c_i	Sensitivity coefficient	$c_i \cong \frac{\partial f}{\partial x_i}$
	$u_i(y)$	Contribution in terms of uncertainty of the variable xi to the result of the measure	$u_i(y)=\lvert c_i \rvert \cdot u(x_i)$
The combined standard uncertainty of the measurement result	$u(y)$	The combined standard uncertainty of the measurement result	$u(y) = \sqrt{\sum_1^N u_i^2(y)}$
The expanded uncertainty of the measurement result	$U(y)$	Expanded uncertainty	$U(y) = k \cdot u(y)$
	k	Coverage factor	k = 2

<u>Presentation n°1 of the uncertainty budget</u>

Input quantity X_i	Estimation x_i	Standard Uncertainty $u(x_i)$	Probability Distribution	Sensitivity Coefficient C_i	Uncertainty Contribution $u_i(y)=\lvert C_i \rvert \cdot u(x_i)$
X_1	x_1	$u(x_1)$		C_1	$u_1(y)=\lvert C_1 \rvert \cdot u(x_1)$
X_2	x_2	$u(x_2)$		C_2	$u_2(y)=\lvert C_2 \rvert \cdot u(x_2)$
:	:	:		:	:
X_N	x_N	$u(x_N)$		C_N	$u_N(y)=\lvert C_N \rvert \cdot u(x_N)$
Y	y				$u(y)$

Presentation n°2 of the uncertainty budget

Origin	Standard Uncertainty $u(x_i)$	Probability Distribution	Sensitivity Coefficient λ_i	Uncertainty Contribution
1: MESURAND Variation V1 Variation V2	 u_1 u_2		 λ_1 λ_2	 $\lvert\lambda_1\rvert.u_1$ $\lvert\lambda_2\rvert.u_2$
2: MEASURING INSTRUMENT I1 : I2 : I3:	 u_3 u_4 u_5		 λ_3 λ_4 λ_5	 $\lvert\lambda_3\rvert.u_3$ $\lvert\lambda_4\rvert.u_4$ $\lvert\lambda_5\rvert.u_5$
3: MEASUREMENT METHOD M1: M2: M3:	 u_6 u_7 u_8		 λ_6 λ_7 λ_8	 $\lvert\lambda_6\rvert.u_6$ $\lvert\lambda_7\rvert.u_7$ $\lvert\lambda_8\rvert.u_8$
4: INFLUENCE QUANTITIES G1 G2 G3	 u_9 u_{10} u_{11}		 λ_9 λ_{10} λ_{11}	 $\lvert\lambda_9\rvert.u_9$ $\lvert\lambda_{10}\rvert.u_{10}$ $\lvert\lambda_{11}\rvert.u_{11}$
5: OPERATORS	u_{12}		λ_{12}	$\lvert\lambda_{12}\rvert.u_{12}$
GLOBAL UNCERTAINTY (Algebraic Sum of the corrections)				$u = \sqrt{\sum_j \lambda_j^2 u_j^2}$

5.6.1 When the input quantities are correlated

When the input quantities are correlated, the appropriate expression for the combined variance $u_C^2(y)$ is:

$$u_C^2(y) = \sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}u(x_i,x_j) = \sum_{i=1}^{N}\left(\frac{\partial f}{\partial x_i}\right)^2 u^2(x_i) + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}u(x_i,x_j)$$

Where *xi* et *xj* are the estimates of X_i et X_j and $u(x_i,x_j) = u(x_j,x_i)$ is the estimated covariance associated with x_i and x_j.

The degree of correlation between xi and xj is characterised by **the correlation coefficient**:

$$r(x_i,x_j) = \frac{u(x_i,x_j)}{u(x_i)u(x_j)}$$

Where $r(x_i, x_j) = r(x_j, x_i)$ and $-1 \leq r(x_i, x_j) \leq +1$

If the estimates *xi* and *xj* are independent, $r(x_i, x_j) = 0$

So,

$$u_C^2(y) = \sum_{i=1}^{N} c_i^2 u^2(x_i) + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} c_i c_j u(x_i) u(x_j) r(x_i, x_j)$$

5.6.2 Value of the covariance of two variables with two arithmetic means

Consider two arithmetic means q and r, which estimate the mathematical μ_q and μ_r of two random quantities q and r, and assume that q and r are computed from n independent pairs of simultaneous observations of q and r made in the same measurement conditions.

Then, the covariance of q and r is estimated by:

$$s(\bar{q}, \bar{r}) = \frac{1}{n(n-1)} \sum_{k=1}^{n} (q_k - \bar{q})(r_k - \bar{r})$$

And

$$r(x_i, x_j) = r(\bar{X}_i, \bar{X}_j) = s(\bar{X}_i, \bar{X}_j) / [s(\bar{X}_i) s(\bar{X}_j)]$$

5.7 Determination of expanded uncertainty

5.7.1 Expanded uncertainty

The additional measure of uncertainty that meets the requirement of providing an interval of the kind indicated is termed *expanded uncertainty* and is denoted by *U*.

The expanded uncertainty *U* is obtained by multiplying the composite standard uncertainty $u_C(y)$ by *a coverage factor k*:

$$U = k u_C(y)$$

The result of a measurement is then conveniently expressed as Y = y ± U, which is interpreted to mean that the best estimate of the value attributable to the measurand Y is y, and that y - U to y + U is an interval that may be expected to encompass a large fraction of the distribution of values that could reasonably be attributed to Y.

Such an interval is also expressed by $y - U \leq Y \leq y + U$

This interval is called «**intervalle de confiance »** or « **niveau de confiance »** in French.

This corresponds in English to **"confidence level"** or **"level of confidence".**

5.7.2 Choosing a coverage factor

The value of the coverage factor k is chosen on the basis of the level of confidence required of the interval y − U to y + U. In general, k will be in the range **2 to 3.**

However, for special applications, *k* may be outside this range. Extensive experience with and full knowledge of the uses to which a measurement result will be put can facilitate the selection of a proper value of k.

Occasionally, one may find that a known correction b for a systematic effect has not been applied to the reported result of a measurement, but instead an attempt is made to take the effect into account by enlarging the “uncertainty” assigned to the result. This should be avoided; only in very special circumstances should corrections for known significant systematic effects not be applied to the result of a measurement.

5.7.3 Degrees of freedom and confidence levels

The expanded uncertainty $U_p = k_p u_c(y)$ is obtained from the estimate y of the measurand Y, and from the combined standard uncertainty $u_c(y)$ of this estimate. From this expanded uncertainty, we define an interval $y - U_p < Y < y + U_p$, which corresponds to a probability or to a confidence level p that are specified and high in number.

The Coverage factor k_p produces, around the measurement result y, a range that can be expected to include a large, specified fraction p of the distribution of values that could be reasonably attributed to the measurand Y.

For example, for a quantity z described by a normal distribution with expectation z and standard deviation σ, the value of k_p that produces an interval $\mu_z \pm k_p\sigma$ that encompasses the fraction *p* of the distribution, and thus has a coverage probability or level of confidence *p*.

Level of confidence p (percentage)	**Coverage factor** k_p
68.27	1
90	1.645
95	1.960
95.45	2
99	2.576
99.73	3

<u>The **t-distribution or Student's distribution** and the degrees of freedom</u>

According to the law of t and degrees of freedom, we have:

$$U_p = k_p u_c(y) = t_p(\nu) u_c(y)$$

And

$$k_p = t_p(\nu)$$

Effective number of degrees of freedom

The distribution of that variable may be approximated by a t-distribution with an effective degrees of freedom v_{eff} obtained by the formula of Welch-Satterthwaite.

$$v_{eff} = \frac{u_c^4(y)}{\sum_{i=1}^{N} \frac{u_i^4(y)}{v_i}}$$

Determination of the number of degrees of freedom case of a type A method

The number of degrees of freedom *v* is equal to *n - 1* for a single quantity estimated by the arithmetic mean of *n* independent observations.

If the *n* independent observations are used to determine both the slope and intercept of a straight line by the method of least squares, the degrees of freedom of their respective standard uncertainties is *v = n − 2*. For a least-squares fit of *m parameters* to *n* data points, the degrees of freedom of the standard uncertainty of each parameter is *v = n – m.*

Determination of the number of degrees of freedom case of a type B method

For a Type B evaluation of standard uncertainty, it is a subjective quantity whose value is obtained by scientific judgement based on the pool of available information.

$$v_i \approx \frac{1}{2}\frac{u^2(x_i)}{\sigma^2[u(x_i)]} = \frac{1}{2}\left[\frac{\Delta u(x_i)}{u(x_i)}\right]^{-2}$$

The degrees of freedom associated with a standard uncertainty u (xi), obtained from a Type B evaluation with lower and upper limits a- and a +, are defined in such a way that the probability that the quantity in question lies outside these limits is extremely small, degrees of freedom can be considered $\nu_i = \infty$

Example: Consider that one's knowledge of how input estimate xi was determined and how its standard uncertainty u(xi) was evaluated leads one to judge that the value of u(xi) is reliable to about 25 percent. This may be taken to mean that the relative uncertainty is Δu(xi)/u(xi) = 0.25, and thus from the above equation, vi = $(0{,}25)^{-2}/2 = 8$. If instead one had judged the value of u(xi) to be reliable to only about 50 percent, then $v_i = 2$

Depending on the needs of the potential users of a measurement result, it may be useful, in addition to v_{eff},, to calculate and also give the values of v_{effA} et v_{effB}, by treating separately the standard uncertainties obtained from the Type A and Type B evaluations.
If the contributions to $u_c^2(y)$ of the Type A and Type B standard uncertainties alone are denoted, respectively, by $u_{cA}^2(y)$ and $u_{cB}^2(y)$, the various quantities are related by:

$$u_c^2(y) = u_{cA}^2(y) + u_{cB}^2(y)$$

And

$$\frac{u_c^4(y)}{v_{eff}} = \frac{u_{cA}^4(y)}{v_{effA}} + \frac{u_{cB}^4(y)}{v_{effB}}$$

Example: Consider that $Y = f(X_1, X_2, X_3) = bX_1X_2X_3$ and that the estimated inputs x_1, x_2, x_3 are normally distributed, their estimates X_1, X_2, X_3 are respectively the arithmetic means of $n_1 = 10$,

$n_2 = 5$, et $n_3 = 15$ independent repeat observations, with relative standard uncertainties $\frac{u(x_1)}{x_1} = 0.25$ percent, $\frac{u(x_2)}{x_2} = 0.57$ percent, and $\frac{u(x_3)}{x_3} = 0.82$ percent. In this case $C_i = \partial f / \partial X_i = Y / X_i$ (to be evaluated at x_1, x_2, x_3).

$$[u_C(y)/y]^2 = \sum_{i=1}^{3} \left(\frac{u(x_i)}{x_i} \right)^2 = (1.03\, percent)^2$$

The Welch-Satterthwaite formula becomes with the relative uncertainties:

$$\nu_{eff} = \frac{[u_C(y)/y]^4}{\sum_{i=1}^{3} \frac{[u(x_i)/x_i]^4}{\nu_i}}$$

We have :

$$\nu_{eff} = \frac{1.03^4}{\frac{0.25^4}{10-1} + \frac{0.57^4}{5-1} + \frac{0.82^4}{15-1}} = 19.0$$

The value of t_p for p = 95 percent and v = 19 is (according to the table) $t_{95}(19) = 2.09$; hence the relative expanded uncertainty for this level of confidence is $U_{95} = 2.09 \times (1.03$ percent) = 2.2 percent.

We can then state that $Y = y \pm U_{95} = y(1 \pm 0.022)$

5.8 Expression of the result of a measurement

When reporting the result of a measurement, and when the measure of uncertainty is the combined standard uncertainty $u_C(y)$, one should:

a) give a full description of how the measurand Y is defined
b) state the result of the measurement as Y = y ± U and give the units of y and U;
c) include the relative expanded uncertainty $U/\lvert y \rvert$, when appropriate, (with the condition $\lvert y \rvert \neq 0$);
d) give the value of k used to obtain U [or, for the convenience of the user of the result, give both k and $u_c(y)$];
e) give the approximate level of confidence associated with the interval y ± U and state how it was determined;
f) provide a detailed report describing how to obtain the result of a measurement and its uncertainty or refer to a published document that includes it

<u>Detailed Report includes the following elements:</u>

a) give the value of each input estimate xi and its standard uncertainty u(xi) together with a description of how they were obtained;
b) give the estimated covariances or estimated correlation coefficients (preferably both) associated with all input estimates that are correlated, and the methods used to obtain them;
c) give the degrees of freedom for the standard uncertainty of each input estimate and how it was obtained;
d) give the functional relationship $Y = f(X_1, X_2, ..., X_N)$ and, when they are deemed useful, the partial derivatives or sensitivity coefficients ∂f/∂xi. However, any such coefficients determined experimentally should be given.

If the detailed report deemed useful for potential users of the measurement result, for example to assist in the subsequent calculation of enlargement factors or to assist in understanding the measurement.

When the measurement of the uncertainty is $u_C(y)$, it is preferable to state the numerical result of the measurement in one of the **following four ways:**

The quantity whose value is being reported is assumed to be a nominally 100 g standard of mass m_S; u_C is defined elsewhere in the document reporting the result:

1) "m_S = 100.021 47 g with (a combined standard uncertainty) u_C = 0.35 mg."

2) "m_S = 100.021 47 (35) g, where the number in parentheses is the numerical value of (the combined standard uncertainty) u_C which relates to the last two corresponding digits of the result provided."

3) "m_S = 100.021 47 (0.000 35) g, where the number in parentheses is the numerical value of (the combined standard uncertainty) u_C expressed with the unit of the result provided."

4) "m_S = (100.021 47 ± 0.000 35) g, where the number following the symbol ± is the numerical value of (the combined standard uncertainty) $u_C$and not a confidence interval."

The ideal is to give uncertainty on the following form:

"m_S = (100.021 47 ± 0.000 79) g, where the number following the symbol ± is the numerical value of (the expanded uncertainty) U = k u_C, with U determined from (the combined standard uncertainty)

The numerical values of the estimate **y** and its standard uncertainty $u_C(y)$ or its expanded uncertainty U shall not be given with an excessive number.

It is usually sufficient to provide $u_C(y_i)$ and U [as well as the standard uncertainties $u(x_i)$ of the input estimates x_i with **two significant digits at the most**, which in some cases may require additional digits. to avoid the propagation of rounding errors in subsequent calculations.

If rounding reduces the numerical value of the measurement uncertainty by more than 5%, rounding to the higher value must be used.

The numerical value of the result of measurement **y** should normally be rounded to the least significant of the value of the expanded uncertainty assigned to the result of the measurement in the final declaration.

A measurement result has four (4) elements:

1) Numeric value with correct number of decimals
2) Unit
3) Expended uncertainty and the level of confidence used to define the range of expanded uncertainty.
4) The coverage factor used (noted k, for ex k = 2)

5.8.1 Determination of the number of significant digits.

In a given number, numbers other than zero are significant. Zeros if they are placed in the heading of the number are not significant.

Examples:

6.8; 2 significant digits
6.80; 3 significant digits
6800; 4 significant digits
0.68; 2 significant digits

5.8.2 Examples of sensitivity coefficients obtained from mathematical models

Function	Sensitivity coefficient
$Y = k_1X_1 + k_2.X_2$	$\lambda_1 = \frac{\partial Y}{\partial X_1} = k_1\,; \lambda_2 = \frac{\partial Y}{\partial X_2} = k_2$
$Y = kX_1X_2$	$\lambda_1 = \frac{\partial Y}{\partial X_1} = kX_2 = \frac{Y}{X_1}; \lambda_2 = \frac{\partial Y}{\partial X_2} = kX_1 = \frac{Y}{X_2}$
$Y = k\frac{X_1}{X_2}$	$\lambda_1 = \frac{\partial Y}{\partial X_1} = \frac{k}{X_2} = \frac{Y}{X_1}; \lambda_2 = \frac{\partial Y}{\partial X_2} = -\frac{kX_1}{X_2^2} = -\frac{Y}{X_2}$
$Y = \frac{k}{X}$	$\lambda = \frac{\partial Y}{\partial X} = -\frac{k}{X^2} = -\frac{Y}{X}$
$Y = kX_1^{\alpha_1}X_2^{\alpha_2}$	$\lambda_1 = \frac{\partial Y}{\partial X_2} = k\alpha_1X_1^{\alpha_1-1}X_2^{\alpha_2} = \alpha_1\frac{Y}{X_1}\,;$ $\lambda_2 = \frac{\partial Y}{\partial X_2} = k\alpha_2X_1^{\alpha_1}X_2^{\alpha_2-1} = \alpha_2\frac{Y}{X_2}\,;$
$Y = kX^2$	$\lambda = \frac{\partial Y}{\partial X} = 2kX = \frac{2Y}{X}$
$Y = \sqrt{X}$	$\lambda = \frac{\partial Y}{\partial X} = \frac{1}{2\sqrt{X}} = \frac{1}{2Y}$
$Y = e^{kX}$	$\lambda = \frac{\partial Y}{\partial X} = ke^{kX} = kY$
$Y = In(X)$	$\lambda = \frac{\partial Y}{\partial X} = \frac{1}{X}$

In stating the final results, it may sometimes be appropriate to round up the uncertainties to the higher number rather than to the nearest figure. For example, $u_C(y)$ = 10.47 mΩ could be rounded to 11 mΩ. However, common sense must prevail and a value such that $u(x_i)$ = 28.05 kHz should be rounded down to 28 kHz.

Input and output estimates should be rounded according to their uncertainties; for example, if y = 10.057 62 Ω with $u_C(y)$ = 27 mΩ, y should be rounded to 10.058 Ω.

Correlation coefficients must be given with three significant digits if their absolute values are close to unity.

5.9 Summary of the procedure for assessing and expressing uncertainty

The steps to evaluate and express the uncertainty of the result of a measurement, as presented in this guide, can be summarised as follows:

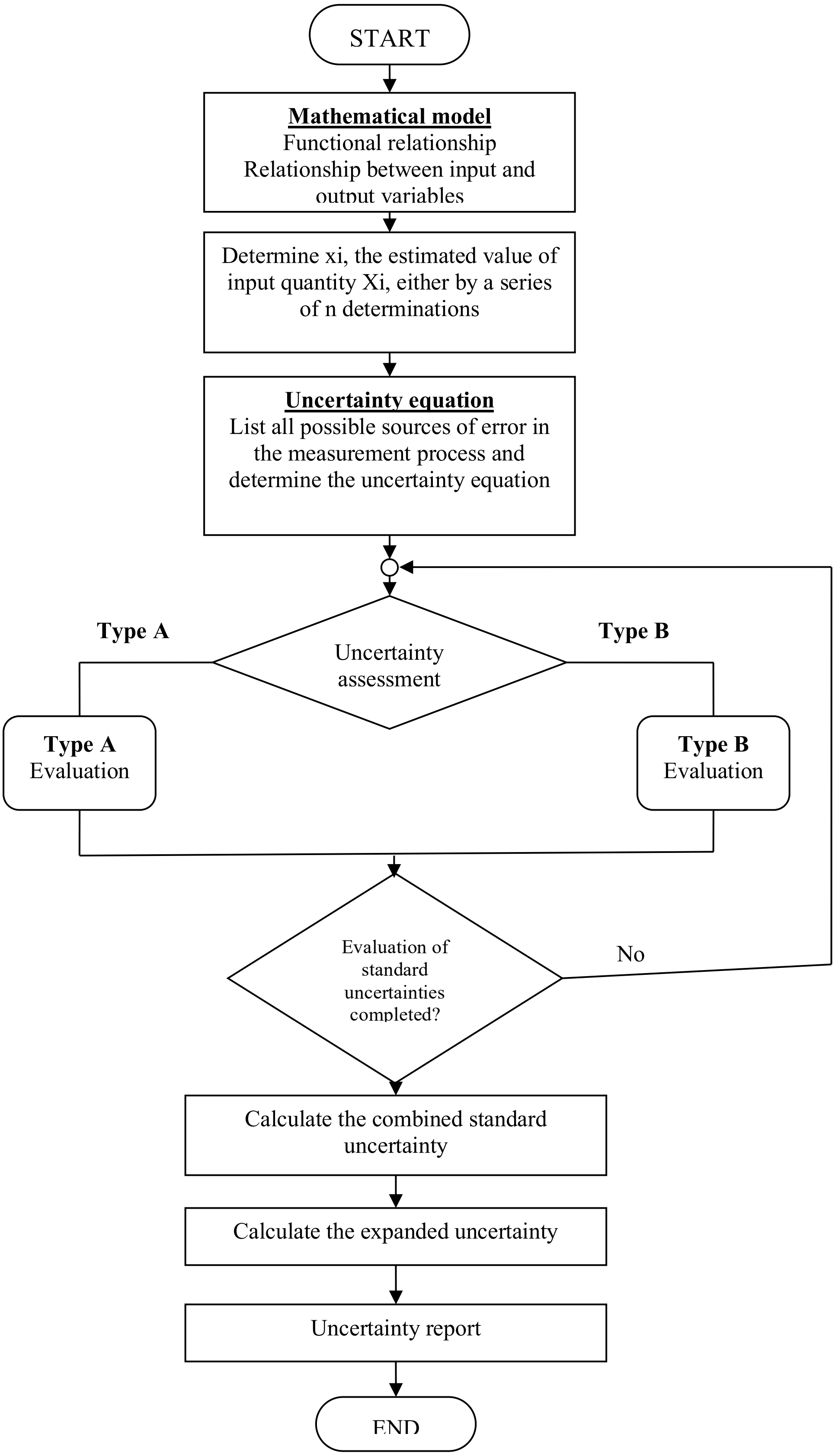
1) Express mathematically the relationship between the measurand Y and the input quantities Xi on which Y depends: $Y = f(X_1, X_2, ..., X_N)$. The function f should contain every quantity, including all corrections and correction factors, that can contribute a significant component of uncertainty to the result of the measurement.
2) Determine *xi*, the estimated value of the input variable *Xi*, either on the basis of the statistical analysis of series of observations, or by other means.
3) Evaluate *the standard uncertainty u(xi)* of each input estimate *xi.* For an input estimate obtained from the statistical analysis of series of observations, carry out a*Type A evaluation* of *standard uncertainty*. For an input estimate obtained by other means, carry out a*Type B evaluation of standard uncertainty.*
4) Evaluate the covariance associated with all input estimates that are correlated.
5) Calculate the result of the measurement, that is, the estimate y of the measurand Y, from the functional relationship f using for the input quantities Xi the estimates xi obtained in step 2.
6) Determine the *combined standard uncertainty* $u_c(y)$ of the measurement result y from the standard uncertainties and covariances associated with the input estimates. If the measurement determines simultaneously more than one output quantity, calculate their covariances.
7) If it is necessary to give an *expanded uncertainty U*, whose purpose is to provide an interval $y - U$ to $y + U$ that may be expected to encompass a large fraction of the distribution of values that could reasonably be attributed to the measurand Y, multiply the combined standard uncertainty $u_c(y)$ by a coverage factor k, typically in the range 2 to 3, to obtain $U = ku_c(y)$. Choose *k* based on the confidence level required for the interval.
8) Report the result of the measurement y together with its combined standard uncertainty $u_c(y)$ or expanded uncertainty *U*. Use one of the formats recommended of expression. Describe how the values of *y* and $u_c(y)$ or *U* were obtained.

Table of values of $t_p(\nu)$ from the t- distribution for degrees of freedom ν that defines an interval $-t_p(\nu)$ to $+t_p(\nu)$ that encompasses the fraction p of the distribution

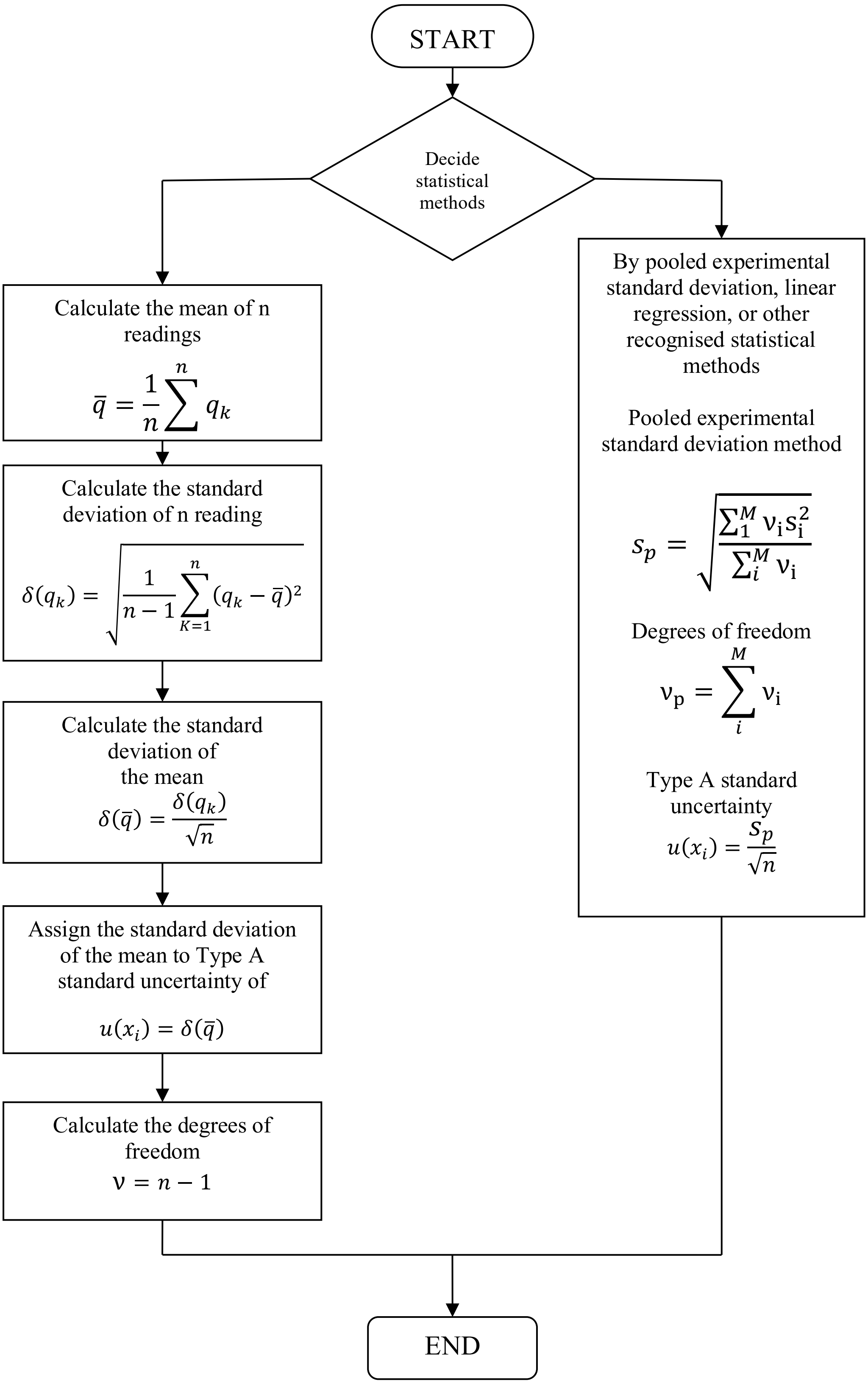
Degrees of Freedom	Fraction p in percent					
ν	68.27[(a)]	90.00	95.00	95.45[(a)]	99.00	99.73[(a)]
1	1.84	6.31	12.71	13.97	63.66	235.8
2	1.32	2.92	4.30	4.53	9.92	19.21
3	1.20	2.35	3.18	3.31	5.84	9.22
4	1.14	2.13	2.78	2.87	4.60	6.62
5	1.11	2.02	2.57	2.65	4.03	5.51
6	1.09	1.94	2.45	2.52	3.71	4.90
7	1.08	1.89	2.36	2.43	3.50	4.53
8	1.07	1.86	2.31	2.37	3.36	4.28
9	1.06	1.83	2.26	2.32	3.25	4.09
10	1.05	1.81	2.23	2.28	3.17	3.96
11	1.05	1.80	2.20	2.25	3.11	3.85
12	1.04	1.78	2.18	2.23	3.05	3.76
13	1.04	1.77	2.16	2.21	3.01	3.69
14	1.04	1.76	2.14	2.20	2.98	3.64
15	1.03	1.75	2.13	2.18	2.95	3.59
16	1.03	1.75	2.12	2.17	2.92	3.54
17	1.03	1.74	2.11	2.16	2.90	3.51
18	1.03	1.73	2.10	2.15	2.88	3.48
19	1.03	1.73	2.09	2.14	2.86	3.45
20	1.03	1.72	2.09	2.13	2.85	3.42
25	1.02	1.71	2.06	2.11	2.79	3.33
30	1.02	1.70	2.04	2.09	2.75	3.27
35	1.01	1.70	2.03	2.07	2.72	3.23
40	1.01	1.68	2.02	2.06	2.70	3.20
45	1.01	1.68	2.01	2.06	2.69	3.18
50	1.01	1.68	2.01	2.05	2.68	3.16
100	1.005	1.660	1.984	2.025	2.626	3.077
∞	1.000	1.645	1.960	2.000	2.576	3.000

a) For a quantity z described by a normal distribution with expectation μ z and standard deviation σ, the interval μ z ± kσ encompasses p = 68.27 percent, 95,45 percent and 99,73 percent of the distribution for k = 1, 2 and 3, respectively.

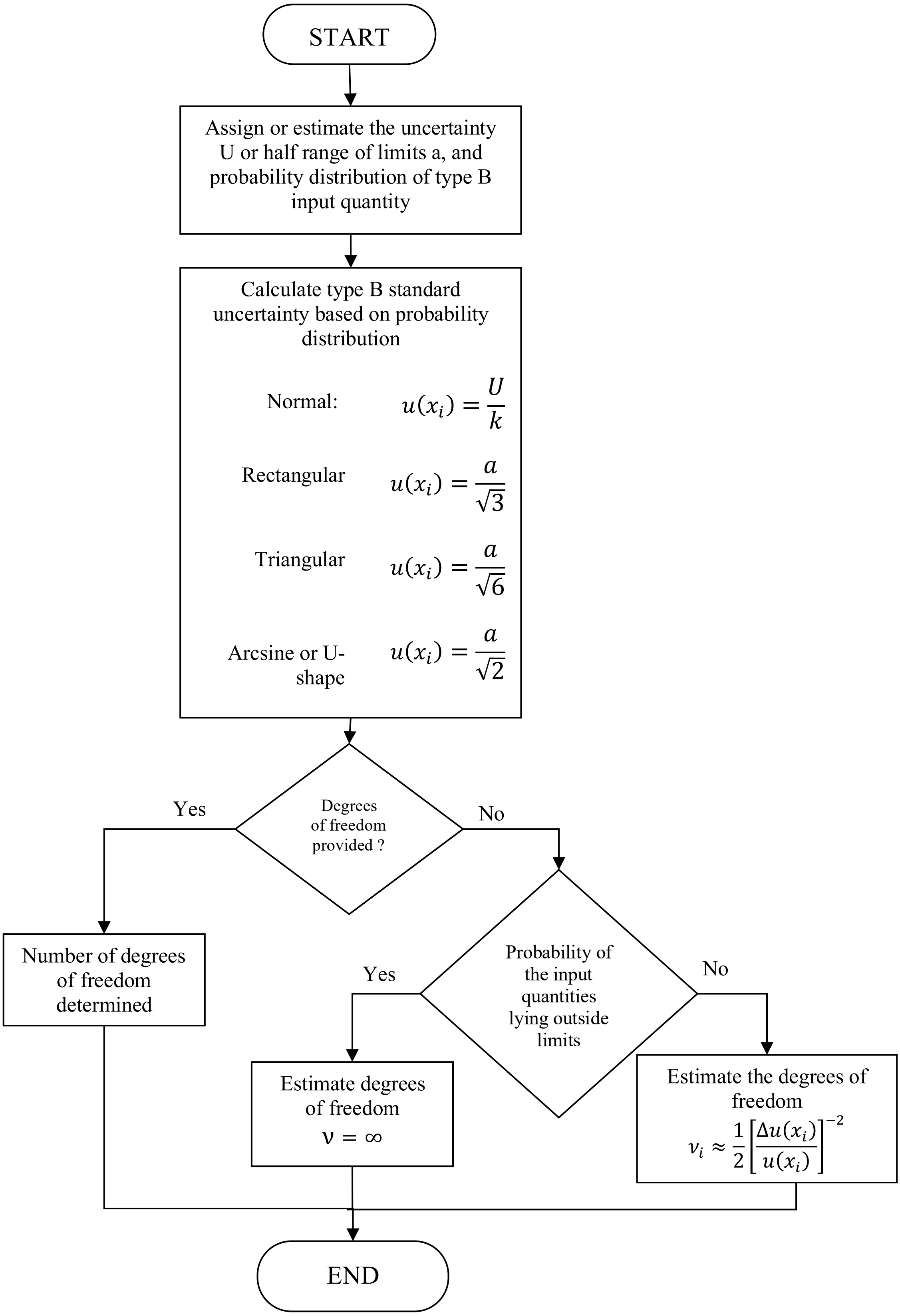
FLOWCHART OF SUMMARY FOR EVALUATING UNCERTAINTY OF MEASUREMENT



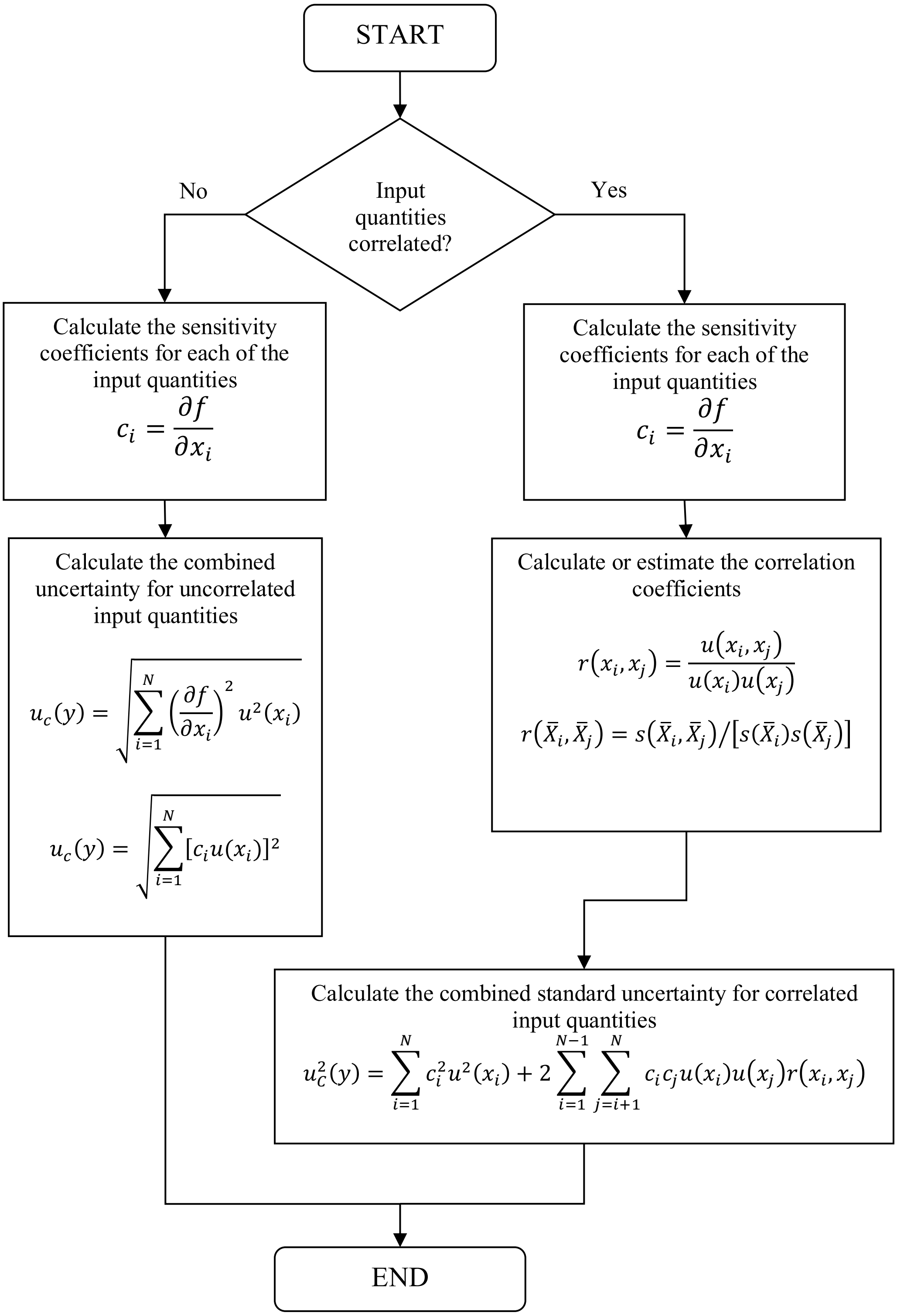
FLOWCHART FOR EVALUATING TYPE A STANDARD UNCERTAINTY



FLOWCHART FOR EVALUATING TYPE B STANDARD UNCERTAINTY



FLOWCHART FOR EVALUATING TYPE B STANDARD UNCERTAINTY



FLOWCHART FOR CALCULATING EXPANDED UNCERTAINTY

START

↓

Calculate the effective degrees of freedom of the combined standard uncertainty

$$\nu_{eff} = \frac{u_c^4(y)}{\sum_{i=1}^{N} \frac{u_i^4(y)}{\nu_i}}$$

↓

Obtain the coverage factor **k** from the "t" distribution table at the appropriate degrees of freedom and level of confidence.

↓

Calculate the expanded uncertainty

$$U = k u_c(y)$$

↓

END

6 EXAMPLES OF EVALUATION OF UNCERTAINTY OF MEASUREMENT

EXAMPLE 1: DETERMINATION OF SENSITIVITY COEFFICIENTS OF THE FOLLOWING FUNCTIONS

When a potential difference V is applied to the terminals of resistance whose value depends on the temperature, on the resistance R_0 at the defined temperature t_0 and on the linear coefficient of temperature α, the power P (the measurand) dissipated by the resistor at the temperature *t* depends on *V, R_0, α*, and *t* according to the following formula:

$$P = f(V, R_0, \alpha, t) = V^2/\{R_0[1 + \alpha(t - t_0)]\}$$

$$\frac{\partial P}{\partial V} = 2V/\{R_0[1 + \alpha(t - t_0)]\} = 2P/V$$

$$\frac{\partial P}{\partial R_0} = -V^2/\{R_0^2[1 + \alpha(t - t_0)]\} = -P/R_0$$

$$\frac{\partial P}{\partial \alpha} = -V^2(t - t_0)/\{R_0[1 + \alpha(t - t_0)]^2\} = -P(t - t_0)/[1 + \alpha(t - t_0)]$$

$$\frac{\partial P}{\partial t} = -V^2\alpha/\{R_0[1 + \alpha(t - t_0)]^2\} = -P\alpha/[1 + \alpha(t - t_0)]$$

When measuring the magnification of an afocal telescope, we note the following positions:

- on the graduated bench in mm
- on the Vernier of the graduated viewfinder in 1/10 mm

x_1 : the first position, located on the bench of the object AB, position remote from L
and x'_1 : the position of the rack of the viewfinder aiming A'B '.

x_2 : the a new position of the object, close to L and still marked by direct reading on the bench and x'_2 : the new position of the viewfinder rack, which has not moved on the bench.

The magnification G is calculated with the following formula:

$$G = f(x_1, x_2, x'_1, x'_2) = \sqrt{\frac{x_1 - x_2}{x'_1 - x'_2}}$$

$$u^2(G) = \left(\frac{\partial G}{\partial x_1}\right)^2 u^2(x_1) + \left(\frac{\partial G}{\partial x_2}\right)^2 u^2(x_2) + \left(\frac{\partial G}{\partial x'_1}\right)^2 u^2(x'_1) + \left(\frac{\partial G}{\partial x'_2}\right)^2 u^2(x'_2)$$

$$\frac{\partial G}{\partial x_1} = \frac{1}{2G} \cdot \frac{1}{(x'_1 - x'_2)}$$

$$\frac{\partial G}{\partial x_2} = \frac{1}{2G} \cdot \frac{-1}{(x_1' - x_2')}$$

$$\frac{\partial G}{\partial x_1'} = -\frac{1}{2} \cdot \frac{G}{(x_1' - x_2')}$$

$$\frac{\partial G}{\partial x_2'} = \frac{1}{2} \cdot \frac{G}{(x_1' - x_2')}$$

EXAMPLE 2: CALIBRATION OF A MASS

Determination of sensitivity coefficients, combined standard uncertainty, expanded uncertainty and uncertainty budget when calibrating a mass.

In the context of our application, s_x is determined during the evaluation of the comparator by carrying out a repeatability test on 10 series of 4 weightings (ABBA) of a load simulating the weight to be calibrated.

Determination of uncertainty according to the two types of evaluation of measurement uncertainty methods:

- **Type A methods:** those evaluated according to a statistical method

- **Type B methods:** those evaluated by other methods.

Type A components

The uncertainty of repeatability u_A corresponds to the standard deviation of the n determinations:

$$u_A = \frac{s_x}{\sqrt{n}}$$

with $s_x = \sqrt{\frac{\sum_i (x_i - \bar{x})}{n-1}}$

To take into account a possible drift of the comparator, s_x is replaced by $s_x(\max())$, the estimate of $s_x(\max())$. Therefore:

$$u_A = \frac{s_x(\max())}{\sqrt{n}}$$

where n is the number of measurement cycles performed during the calibration process.

Type B components

Standard uncertainty due to the resolution u_{B1}

The standard uncertainty due to the resolution of the comparator for a rectangular distribution is:

$$u_{B1} = \frac{d/2}{\sqrt{3}}$$

Since calibration is done with two masses (the mass standard and the mass under calibration), the standard uncertainty on the resolution is counted twice, which is equivalent to a multiplication factor of $\sqrt{2}$.

$$u_{B1} = \frac{d/2}{\sqrt{3}} \cdot \sqrt{2} = \frac{d}{\sqrt{6}}$$

The standard uncertainty of the mass standard u_{B2}

The calibration uncertainty of the mass standard is obtained from its calibration certificate. The calibration certificate gives the calibration uncertainty $U_E$of the standard weight with a coverage factor k equal to 2. Therefore, the standard uncertainty of calibration is:

$$u_{B2} = \frac{U_E}{k}$$

When several standards are used in the comparison frame, the calibration uncertainty U_E of the calibration is equal to the algebraic sum of the calibration uncertainty of the standards used:

$$U_E = \sum_i U_{Ei}$$

The standard uncertainty of the drift of the mass standard u_{B3}

Two situations can arise:

If no information is available on previous calibrations or if the standard is at its first calibration then the standard uncertainty of the drift can be calculated from its calibration uncertainty:

$$u_{B3} = u_{B2} = \frac{U_E}{2}$$

If information on previous calibrations is available, the uncertainty of drift can also be calculated from the largest difference in mass between two calibrations ΔE:

$$u_p = \frac{|\Delta E|}{\sqrt{3}}$$

The standard uncertainty of drift is:

$$u_{B3} = max(u_{B2}, u_p)$$

The standard uncertainty due to the density of the air u_{B4} ;

The formula for the determination of the density of air is given by the formula of the International Committee of Weights and Measures (CIPM) of 1981/1991:

$$\rho_a = \frac{p \cdot M_a}{ZRT}\left[1 - x_v\left(1 - \frac{M_v}{M_a}\right)\right]$$

Where p = *The pressure*
M_a = *The molar mass of dry air*
Z = *The Compressibility factor*
R = *The molar gas constant*
T = *The thermodynamic temperature*
x_v = *The mole fraction of water vapour*
M_v = *The molar mass of water*

The most accurate and simplified formula for determinining the air density is the following:

The most accurate simplified formula for determining air density is given in Appendix E of R111:

$$\rho_a = \frac{0.34848 \cdot p - 0.009 \cdot hr \cdot exp(0.061 \cdot t)}{273.15 + t} \text{ (kg/m}^3\text{)} \quad \textbf{(15)}$$

with a relative uncertainty of 2.10^{-4} $(\frac{\Delta\rho_a}{\rho_a})$

Where p *is expressed in hPa ;* $900hPa < p < 1100hPa$
hr *is expressed in percent (%) example 40 pour 40% ;* $hr < 80\%$
t *is expressed in °C;* $10°C < t < 30°C$

Let us apply the law of propagation of uncertainties to the density of air:

$$\rho_a = f(p, t, hr)$$

$$u_c^2(\rho_a) = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial x_i}\right)^2 \cdot u^2(x_i) + 2\sum_{i=1}^{N} \sum_{j=i+1}^{N} \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} u(x_i, x_j)$$

As the estimates of x_i et x_j are independent, the correlation coefficients are equal to zero and therefore the covariances are also equal to zero and the equation is reduced to the following expression:

$$u_c^2(\rho_a) = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial x_i}\right)^2 \cdot u^2(x_i)$$

Determination of the different sensitivity coefficients:

$$c_i = \frac{\partial f}{\partial x_i}$$

$$\frac{\partial \rho_a}{\partial p} = \frac{0.34848}{273.15 + t}$$

$$\frac{\partial \rho_a}{\partial t} = \frac{-0.34848 \cdot p + 0.009 \cdot hr \cdot [1 - 0{,}061 \cdot (273.15 + t)] \cdot exp(0.061 \cdot t) \cdot}{(273.15 + t)^2}$$

$$\frac{\partial \rho_a}{\partial hr} = -\frac{0.009 \cdot exp(0.061 \cdot t)}{273.15 + t}$$

Using the average values: t= 20°C; p=1013 hPa et hr=50% we obtain:

$$\frac{\partial \rho_a}{\partial p} = 1.2 \cdot 10^{-3}$$

$$\frac{\partial \rho_a}{\partial t} = -4.4 \cdot 10^{-3}$$

$$\frac{\partial \rho_a}{\partial hr} = -1 \cdot 10^{-4}$$

$$u_c^2(\rho_a) = u^2(formula) + (1.2 \cdot 10^{-3})^2 \cdot u^2(p) + (4.4 \cdot 10^{-3}) \cdot u^2(t) + (1 \cdot 10^{-4})^2 \cdot u^2(hr)$$

With

$$u(formula) = \rho_a \cdot 2 \cdot 10^{-4}$$

Standard uncertainty due to the density of the mass standard u_{B5} ;

For mass standard of class E2, F1, F2, M1, M2 and M3, the values of the density are given by OIML recommendation R111 shown in the table below.

The standard uncertainty due to the density of the mass under calibration u_{B6} ;

For mass of class E2, F1, F2, M1, M2 and M3, the values of the density are also given by OIML recommendation R111 in the table below.

Alloy/material	**Assumed density kg/m^3**	**Standard uncertainty (k=2) kg/m^3**
Platinum	21 400	150
Nickel silver	8 600	170
Brass	8 400	170
Stainless steel	7 950	140
Iron	7 800	200
Carbon steel	7 700	200
Cast iron (white)	7 700	400
Cast iron (grey)	7 100	600
Aluminium	2 700	130

First method of estimating of the combined standards uncertainty

Determination of the sensitivity coefficients of the following function $\boldsymbol{M_c = f(X, E_c, \rho_a, \rho_m, \rho_e)}$

$$M_c = X + E_c \cdot \left(1 + (\rho_a - \rho_{ao}) \cdot \left(\frac{1}{\rho_m} - \frac{1}{\rho_e}\right)\right)$$

$$\frac{\partial M_c}{\partial X} = 1$$

$$\frac{\partial M_c}{\partial E_c} = 1$$

$$\frac{\partial M_c}{\partial \rho_a} = \left(\frac{1}{\rho_m} - \frac{1}{\rho_e}\right) \cdot E_c$$

$$\frac{\partial M_c}{\partial \rho_e} = \left(\frac{\rho_a - \rho_{ao}}{\rho_e^2}\right) \cdot E_c$$

$$\frac{\partial M_c}{\partial \rho_m} = -\left(\frac{\rho_a - \rho_{ao}}{\rho_m^2}\right) \cdot E_c$$

The combined standards uncertainty is obtained by applying the law of propagation of the uncertainties while considering that the variables are uncorrelated.

$$u_c^2(M_c) = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial x_i}\right)^2 \cdot u^2(x_i)$$

$$u_c(M_c) = \sqrt{\left(\frac{\partial M_c}{\partial X}\right)^2 \cdot u^2(X) + \left(\frac{\partial M_c}{\partial E_c}\right)^2 \cdot u^2(E_c) + \left(\frac{\partial M_c}{\partial \rho_a}\right)^2 \cdot u^2(\rho_a) + \left(\frac{\partial M_c}{\partial \rho_e}\right)^2 \cdot u^2(\rho_e) + \left(\frac{\partial M_c}{\partial \rho_m}\right)^2 \cdot u^2(\rho_m)}$$

Determination of the expanded uncertainty

The expanded uncertainty is determined by multiplying the combined standards uncertainty by the coverage factor k = 2.

$$U = \pm 2 \cdot u_c$$

Summary table of the uncertainty budget

Uncertainty component	Symbol	Standard uncertainty		Sensitivity coefficient	Standard uncertainty Contribution
		Type A	Type B		
Repeatability of the mean of n determinations	u_A	$u_A = \frac{s_x}{\sqrt{n}}$		1	
Resolution of the comparator	u_{B1}		$u_{B1} = \frac{d}{\sqrt{6}}$	1	
Calibration of the reference standards	u_{B2}		$u_{B2} = \frac{U_E}{k}$	1	
Drift of the reference standards	u_{B3}		$u_{B3} = max(u_{B2}, u_p)$	1	
Density of air	$\boldsymbol{u_{B4}}$		$u(\rho_a)$	$\left(\frac{1}{\rho_m} - \frac{1}{\rho_e}\right) \cdot E_c$	
Density of the reference standards	$\boldsymbol{u_{B5}}$		$u(\rho_e)$	$\left(\frac{\rho_a - \rho_{ao}}{\rho_e^2}\right) \cdot E_c$	
Density of the mass under test	$\boldsymbol{u_{B6}}$		$u(\rho_m)$	$-\left(\frac{\rho_a - \rho_{ao}}{\rho_m^2}\right) \cdot E_c$	
Uncertainty - composite type is determined by calculating the quadratic sum of the contributions of each standards uncertainty component: $u_c = \sqrt{\sum_i u_i^2}$					
The expanded uncertainty is determined by multiplying the combined standards uncertainty by the expansio factor k = 2: $U = \pm 2u_c$					

Second method of calculation

This method allows the evaluation of the correction due to the air buoyancy as well as its uncertainty of measurement.

The conventional mass under calibration is:

$$M_c = x + E_c + \left((\rho_a - \rho_{ao}) \cdot \left(\frac{1}{\rho_m} - \frac{1}{\rho_e}\right)\right) \cdot E_c$$

The correction due to the air buoyancy is given by the following formula:

$$C = (\rho_a - \rho_{ao}) \cdot \left(\frac{1}{\rho_m} - \frac{1}{\rho_e}\right) \cdot E_c$$

where

ρ_a: is the density of air;
ρ_{ao}: the reference density of the air: 1.2 kg/m^3
ρ_e : density of the reference standards;
ρ_m : density of the mass under test.
E_c : the conventional mass of the reference standard that can be replaced by the nominal mass of the reference standard

Determining the uncertainty of the air buoyancy correction

When applying the law of propagation of the uncertainties by supposing that the variables are uncorrelated:

$$u_c^2(C) = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial x_i}\right)^2 \cdot u^2(x_i) = \left(\frac{\partial C}{\partial \rho_a}\right)^2 \cdot u^2(\rho_a) + \left(\frac{\partial C}{\partial \rho_e}\right)^2 \cdot u^2(\rho_e) + \left(\frac{\partial C}{\partial \rho_m}\right)^2 \cdot u^2(\rho_m)$$

$$\frac{\partial C}{\partial \rho_a} = E_c \cdot \left(\frac{\rho_e - \rho_m}{\rho_e \cdot \rho_m}\right)$$

$$\frac{\partial C}{\partial \rho_m} = -E_c \cdot \left(\frac{\rho_a - \rho_{ao}}{\rho_m^2}\right)$$

$$\frac{\partial C}{\partial \rho_e} = E_c \cdot \left(\frac{\rho_a - \rho_{ao}}{\rho_e^2}\right)$$

$$u_c^2(C) = \left[E_c \cdot \left(\frac{\rho_e - \rho_m}{\rho_e \cdot \rho_m}\right) \cdot u(\rho_a)\right]^2 + [E_c \cdot (\rho_a - \rho_{ao})]^2 \cdot \left(\frac{u^2(\rho_e)}{\rho_e^4} + \frac{u^2(\rho_m)}{\rho_m^4}\right)$$

When we consider that the variables are correlated, we obtain according to R111 the following formula:

$$u_c^2(C) = \left[E_c \cdot \left(\frac{\rho_e - \rho_m}{\rho_e \cdot \rho_m}\right) \cdot u(\rho_a)\right]^2 + [E_c \cdot (\rho_a - \rho_{ao})]^2 \cdot \frac{u^2(\rho_m)}{\rho_m^2} - E_c^2(\rho_a - \rho_{ao}) \cdot [(\rho_a - \rho_{ao}) + 2(\rho_{a1} - \rho_a)] \cdot \frac{u^2(\rho_e)}{\rho_e^2}$$

In these different expressions, $\boldsymbol{E_c}$ can be replaced by the nominal value of the standard $\boldsymbol{M_o}$

Determination of the density of the standard $\boldsymbol{\rho_e}$ and its uncertainty $\boldsymbol{u(\rho_e)}$

The value of the density of the standard depends on the material in which the standard was made. Its density and its standard uncertainty with a coverage factor k = 2 are given in the table given above by R111.

Example: if the standard is in stainless steel, its density and standard uncertainty are:

$$\boldsymbol{\rho_e} = \textbf{7950 kg/m}^3$$

$$\boldsymbol{u(\rho_e) = \frac{U_{\rho e}}{2} = \frac{\pm 140}{2}} = \textbf{±70 kg/m}^3$$

Determination of the density of the mass to be calibrated $\boldsymbol{\rho_m}$ and its uncertainty $\boldsymbol{u(\rho_m)}$

For mass of class E2, F1, F2, M1, M2 and M3, the values are given by the OIML recommendation R111 in the table below.

Alloy/material	Assumed density kg/m³	Standard uncertainty (k=2) kg/m³
Platinum	21 400	150
Nickel silver	8 600	170
Brass	8 400	170
Stainless steel	7 950	140
Iron	7 800	200
Carbon steel	7 700	200
Cast iron (white)	7 700	400
Cast iron (grey)	7 100	600
Aluminium	2 700	130

From these different values and from the value of the nominal mass of the reference standard, we determine:

- **The correction due to the air buoyancy correction C;**
- **The uncertainty on this correction $\boldsymbol{u_c(C)}$.**

The combined standard uncertainty on the conventional mass from this method is obtained as follows:

$$M_c = x + E_c + C$$

With

$$C = \left((\rho_a - \rho_{ao}) \cdot \left(\frac{1}{\rho_m} - \frac{1}{\rho_e}\right)\right) \cdot E_c$$

The combined standard uncertainty is obtained by applying the law of propagation of the uncertainties while admitting that the variables are uncorrelated.

$$u_c^2(M_c) = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial x_i}\right)^2 \cdot u^2(x_i)$$

$$u_c(M_c) = \sqrt{\left(\frac{\partial M_c}{\partial X}\right)^2 \cdot u^2(X) + \left(\frac{\partial M_c}{\partial E_c}\right)^2 \cdot u^2(E_c) + \left(\frac{\partial M_c}{\partial C}\right)^2 \cdot u^2(C)}$$

Then :

$$\frac{\partial M_c}{\partial X} = \frac{\partial M_c}{\partial E_c} = \frac{\partial M_c}{\partial C} = 1$$

$$u_c(M_c) = \sqrt{u^2(X) + u^2(E_c) + \cdot\, u^2(C)}$$

$$u_c(M_c) = \sqrt{u_A^2 + u_{B1}^2 + u_{B2}^2 + u_{B3}^2 + u^2(C)}$$

with :

$u_A = \frac{s_x}{\sqrt{n}}$ $u_{B1} = \frac{d}{\sqrt{6}}$ $u_{B2} = \frac{U_E}{k}$ $u_{B3} = max(u_{B2}, u_p)$

$u_p = \frac{|\Delta E|}{\sqrt{3}}$ where ΔE is the value of the drift between two calibrations.

Determination of the expanded uncertainty

The expanded uncertainty is determined by multiplying the combined standard uncertainty by the coverage factor k = 2.

$$U = \pm 2u_c$$

EXAMPLE 3: CALIBRATION OF A MICROVOLUME MEASURING INSTRUMENT

Determination of the average volume of pure water at 20 ° C

$$V_0 = (I_L - I_E) \times \frac{1}{\rho_w - \rho_A} \times \left(1 - \frac{\rho_A}{\rho_B}\right) \times [1 - \gamma(t - t_0)]$$

V_0 : volume at the reference temperature t_0, in mL
I_L : weighing result of the container full of liquid, in g
I_E : weighing result of empty container, in g
ρ_w : density of the liquid, in g / mL, at the temperature t (ºC)
ρ_A : density of air, in g / ml
ρ_B : density of the mass used during the measurement (substitution) or during the calibration of the balance is assumed to be 8,0 g / ml
γ : coefficient of cubic thermal expansion of the material of the instrument being calibrated, in °C^{-1}
t : temperature of liquid used in the calibration, in °C
t_0 : reference temperature, in ºC

Determination of the density of water ρ_w

Formula n°1: The formula given by Tanaka [6] provides a good basis for standardisation:

$$\rho_w = a_5\left[1 - \frac{(t + a_1)^2(t + a_2)}{a_3(t + a_4)}\right]$$

where:

t : Water temperature, in °C
$a1$ = -3.983035 °C
$a2$ = 301.797 °C
$a3$ = 522528.9 $(°C)^2$
$a4$ = 69.34881 °C
$a5$ = 0.999974950 g/mL

The correction due to the air content of the water can be carried out according to the following formula:

$$\Delta\rho = s_0 + s_1 \cdot t$$

t = Water temperature, in °C

s_0 = -4.612×10-6 g/mL
s_1 = 0.106×10-6 g/mL°C

Formula n° 2:

$$\rho_w = \sum_{i=0}^{4} a_i \times t_w^i \qquad = a_0 + a_1 \times t_w + a_2 \times t_w^2 + a_3 \times t_w^3 + a_4 \times t_w^4, \text{ where}$$

a_0 = 999,85308 kg/m³
a_1 = 6,32693 × 10^{-2} $°C^{-1}$ kg/m³
a_2 = –8,523829 × 10^{-3} $°C^{-2}$ kg/m³
a_3 = 6,943248 × 10^{-5} $°C^{-3}$ kg/m³
a_4 = –3,821216 × 10^{-7} $°C^{-4}$ kg/m³
t_w = water temperature in °C

This equation gives the values of the density of the water expressed in kg / m3, to **3** decimal for a temperature range from 5 ° C to 40 ° C.

According to the second formula of the density of water:

Uncertainty of the density of water according to GUM:

The standard uncertainty of the density of water $u(\rho_W)$ is:

$$u(\rho_W) = \sqrt{u^2(formula) + u^2(stability) + c_w^2 \times u^2(t_w)}$$

The uncertainty of the formula and its composition

The density of the water cannot be known with an uncertainty better than **0.015 kg / m3**, due to the uncertainty of the formula and the presence or absence of dissolved gases.

Assuming a rectangular distribution, the standard uncertainty is calculated as follows:

$$u(formula) = (0{,}015)/\sqrt{3}$$

Standard uncertainty of the stability of ρ_w

Assuming a rectangular distribution, the standard uncertainty of the stability of water density is calculated as follows:

$$u(stability) = \left(\rho_{W_{beginning}} - \rho_{W_{end}}\right)/\sqrt{3}$$

Standard uncertainty of water temperature

Assuming a rectangular distribution, the standard uncertainty is determined from the uncertainty mentioned in the calibration certificate of the thermometer, which is associated with its drift and the uncertainty due to the resolution of the instrument:

$$u(t_w) = \sqrt{u_{resolution}^2 + u_{calibration}^2 + u_{drift}^2}$$

The sensitivity coefficient c_W relating to t_w is

$$c_W = \frac{\partial \rho_W}{\partial t_W} = \sum_{i=1}^{4} i \times a_i \times t_W^{i-1} = a_1 + 2 \times a_2 \times t_W + 3 \times a_3 \times t_W^2 + 4 \times a_4 \times t_W^3$$

Determination of the density of the air

The formula for the determination of the density of air is given by the formula of the International Committee of Weights and Measures (CIPM) of 1981/1991:

$$\rho_a = \frac{p \cdot M_a}{ZRT}\left[1 - x_v\left(1 - \frac{M_v}{M_a}\right)\right]$$

Where p = *The pressure*
M_a = *The molar mass of dry air*
Z = *The Compressibility factor*
R = *The molar gas constant*
T = *The thermodynamic temperature*
x_v = *The mole fraction of water vapour*
M_v = *The molar mass of water*

The most accurate simplified formula for determining air density is given in Appendix E of R111:

$\rho_a = \frac{0,34848 \cdot p - 0,009 \cdot hr \cdot exp(0,061 \cdot t)}{273,15 + t}$ *(kg/m^3)*
with a relative uncertainty of 2.10^{-4} ($\frac{\Delta\rho_a}{\rho_a}$)

Where p *is expressed in hPa ;* $900hPa < p < 1100hPa$
hr *is expressed percent (%) example 40 for 40% ;* $hr < 80\%$
t *is expressed in °C;* $10°C < t < 30°C$

Let us apply the law of propagation of uncertainties to the density of air:

$$\rho_a = f(p, t, hr)$$

$$u_c^2(\rho_a) = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial x_i}\right)^2 \cdot u^2(x_i) + 2\sum_{i=1}^{N}\sum_{j=i+1}^{N} \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} u(x_i, x_j)$$

The estimates of x_i and x_j are independent, the correlation coefficients are equal to zero and therefore the covariances are also equal to zero and the equation is reduced to the following expression:

$$u_c^2(\rho_a) = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial x_i}\right)^2 \cdot u^2(x_i)$$

Determination of the different sensitivity coefficients:

$$c_i = \frac{\partial f}{\partial x_i}$$

$$\frac{\partial \rho_a}{\partial p} = \frac{0.34848}{273.15 + t}$$

$$\frac{\partial \rho_a}{\partial t} = \frac{-0.34848 \cdot p + 0.009 \cdot hr \cdot [1 - 0.061 \cdot (273.15 + t)] \cdot exp(0.061 \cdot t) \cdot}{(273.15 + t)^2}$$

$$\frac{\partial \rho_a}{\partial hr} = -\frac{0.009 \cdot exp(0.061 \cdot t)}{273.15 + t}$$

Using the average values: t= 20°C; p=1013 hPa and hr=50% :

$$\frac{\partial \rho_a}{\partial p} = 1.2 \cdot 10^{-3}$$

$$\frac{\partial \rho_a}{\partial t} = -4.4 \cdot 10^{-3}$$

$$\frac{\partial \rho_a}{\partial hr} = -1 \cdot 10^{-4}$$

$$u_c^2(\rho_a) = u^2(formule) + (1{,}2 \cdot 10^{-3})^2 \cdot u^2(p) + (4{,}4 \cdot 10^{-3}) \cdot u^2(t) + (1 \cdot 10^{-4})^2 \cdot u^2(hr)$$

With

$$u(formula) = \rho_a \cdot 2 \cdot 10^{-4}$$

***Standard uncertainty of air parameters** (t_a, p_a, h)*

Assuming a rectangular distribution, the standard uncertainties are determined from the uncertainty mentioned in the thermometer calibration certificate, which is associated with its drift and the uncertainty due to the resolution of the instrument:

$$u(t) = \sqrt{u_{res\_th}^2 + u_{cal\_th}^2 + u_{drift\_th}^2}$$

$$u(p) = \sqrt{u_{res\_press}^2 + u_{cal\_press}^2 + u_{drift\_press}^2}$$

$$u(hr) = \sqrt{u_{res\_hyg}^2 + u_{cal\_hyg}^2 + u_{drift\_hyg}^2}$$

Determination of uncertainty on the volume V_0

Another expression of the volume V_0 is the following formula:

$$V_0 = \frac{m}{\rho_w(t_w) - \rho_A(t_A, p_A, h_r)} \times \left(1 - \frac{\rho_A(t_A, p_A, h_r)}{\rho_B}\right) \times [1 - \gamma(t - t_0)] + \delta V_{men} + \delta V_{evap} + \delta V_{rep}$$

$\boldsymbol{\delta V_{men}}$ Error due to meniscus reading

$\boldsymbol{\delta V_{evap}}$ Error due to evaporation losses

$\boldsymbol{\delta V_{rep}}$ Error due to the repeatability of the measurement

Determination of the different sensitivity coefficients

The formula of V_0 can be put in the following form:

$$V_0 = m \times A \times B \times C + \delta V_{men} + \delta V_{evap} + \delta V_{rep}$$

$$m = I_L - I_E$$

$$A = \frac{1}{\rho_w - \rho_A}$$

$$B = \left(1 - \frac{\rho_A}{\rho_B}\right)$$

$$C = [1 - \gamma(t - t_0)]$$

The results of the calculation of the different sensitivity coefficients are as follows:

Mass:

$$\frac{\partial V_0}{\partial m} = A \times B \times C = \frac{V_0}{m}$$

Water temperature:

$$\frac{\partial V_0}{\partial t} = m \times A \times B \times (-\gamma)$$

Density of the Air:

$$\frac{\partial V_0}{\partial \rho_A} = m \times C \times A \times \left[\frac{1}{\rho_w - \rho_A} \times \left(1 - \frac{\rho_A}{\rho_B}\right) - \frac{1}{\rho_B}\right] = m \times A \times C \times \left(B \times A - \frac{1}{\rho_B}\right)$$

Density of the mass:

$$\frac{\partial V_0}{\partial \rho_B} = m \times A \times C \times \left(\frac{\rho_A}{\rho_B^2}\right)$$

Coefficient of cubic thermal expansion of the material of the instrument under calibration:

$$\frac{\partial V_0}{\partial \gamma} = m \times A \times B \times \left(-(t - t_0)\right)$$

Reading of the meniscus:

$$\frac{\partial V_0}{\partial \delta V_{men}} = 1$$

Evaporation:

$$\frac{\partial V_0}{\partial \delta V_{evap}} = 1$$

Repeatability of the measurement:

$$\frac{\partial V_0}{\partial \delta V_{rep}} = 1$$

The combined standard uncertainty is calculated as follows:

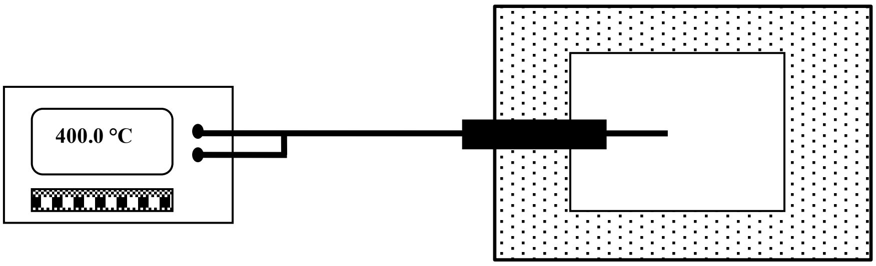
$$u^2(V_0) = \sum_i \left(\frac{\partial V_0}{\partial x_i} \times u(x_i)\right)^2$$

$$u^2(V_0) = \left(\frac{\partial V_0}{\partial m}\right)^2 u^2(m) + \left(\frac{\partial V_0}{\partial t}\right)^2 u^2(t) + \left(\frac{\partial V_0}{\partial \rho_w}\right)^2 u^2(\rho_w) + \left(\frac{\partial V_0}{\partial \rho_A}\right)^2 u^2(\rho_A) + \left(\frac{\partial V_0}{\partial \rho_B}\right)^2 u^2(\rho_B) + \left(\frac{\partial V_0}{\partial \gamma}\right)^2 u^2(\gamma)$$

$$+u^2(\delta V_{men}) + u^2\left(\delta V_{evap}\right) + u^2\left(\delta V_{rep}\right)$$

EXAMPLE 4 TEMPERATURE MEASUREMENT USING A THERMOCOUPLE

A digital thermometer with a Type K thermocouple are used to measure the temperature inside a temperature chamber. The temperature controller of the chamber is set at 400° C (the uncertainty due to thermal leakage is estimated to be negligible).



Installation of the measuring system

Digital thermometer

Resolution: 0.1 ° C
Uncertainty (one year): ± 0.6 ° C

Thermocouple

The K-type thermocouple calibration certificate gives an uncertainty of ± 1.0 ° C with a confidence level of about 95% with a coverage factor k equal to 2.
The correction for the thermocouple at 400 ° C is 0.5 ° C.

Results of measurement record

When the temperature chamber indicator reaches 400 ° C, the readings are taken after a stabilization time of half an hour. Ten (10) measurements are made as shown in the table below:

N°	T (°C)
1	400.1
2	400.0
3	400.1
4	399.9
5	399.9
6	400.0
7	400.1
8	400.2
9	400.0
10	399.9

MATHEMATICAL MODEL

The test temperature t_x is given by:

$$t_x \cong t_r + \Delta t_{tC} + \Delta t_{im} + \Delta t_{der} + \Delta t_{ind} + \Delta t_{res}$$

t_r : temperature reading of the Type K thermocouple.
Δt_{tC} : temperature correction of the Type K thermocouple reading based on its calibration data.
Δt_{im} : temperature correction due to immersion error of the Type K thermocouple.
Δt_{der} : temperature correction due to drift of the Type K thermocouple.
Δt_{ind} : temperature correction due to deviation of the digital thermometer.
Δt_{res} : temperature correction due to the resolution of the digital thermometer.

Evaluation of the measurement uncertainty

$$u(t_x) \cong [u^2(t_r) + u^2(\Delta t_{tC}) + u^2(\Delta t_{im}) + u^2(\Delta t_{der}) + u^2(\Delta t_{ind}) + u^2(\Delta t_{res})]^{1/2}$$

$u(t_r)$: standard uncertainty of the Type K thermocouple reading.
$u(\Delta t_{tC})$: standard uncertainty of the Type K thermocouple correction.
.$u(\Delta t_{im})$: standard uncertainty of the Type K thermocouple immersion correction
$u(\Delta t_{der})$: standard uncertainty of the Type K thermocouple drift correction.
$u(\Delta t_{ind})$: standard uncertainty of the digital thermometer deviation correction.
$u(\Delta t_{res})$: standard uncertainty of the digital thermometer resolution correction

<u>Type A evaluation</u>

Standard uncertainty of the thermocouple reading $u(t_r)$

$$\bar{T} = \frac{1}{10}\sum_{i=1}^{10} T_i$$

$$s(T_i) = \sqrt{\frac{1}{n-1}\sum_{K=1}^{n}(T_i - \bar{T})^2}$$

$$s(T_i) = 0.103\ °C$$

Standard deviation of the mean : $s(\bar{T}) = \frac{s(T_i)}{\sqrt{n}} = \frac{0.103\ °C}{\sqrt{10}} = 0.033\ °C$

Standard uncertainty of the thermocouple correction $u(\Delta t_{tC})$

The correction for the thermocouple reading is 0.5 °C. The standard uncertainty of thermocouple correction $u(\Delta t_{tC})$ is:

$$u(\Delta t_{tC}) = \frac{1.0}{2.0} = 0.5\ °C$$

Standard uncertainty of the thermocouple immersion correction $u(\Delta t_{im})$

The uncertainty limit of the thermocouple immersion correction is ± 0.1 °C.

Assuming a rectangular distribution, standard uncertainty of the thermocouple immersion correction:

$$u(\Delta t_{im}) = \frac{0,1}{\sqrt{3}} = 0.058\ °C$$

Standard uncertainty of the thermocouple drift correction $u(\Delta t_{der})$

The uncertainty limit of the drift is ± 0.2 ° C. Assuming a rectangular distribution, standard uncertainty of the thermocouple drift correction:

$$u(\Delta t_{der}) = \frac{0.2}{\sqrt{3}} = 0.115\ °C$$

Standard uncertainty of the digital thermometer deviation correction Δt_{ind}

From specification, the uncertainty limit of the digital thermometer is ± 0.6 °C. Assuming a rectangular distribution, the standard uncertainty of the digital thermometer deviation correction is:

$$\Delta t_{ind} = \frac{0.6}{\sqrt{3}} = 0.346\ °C$$

Standard uncertainty of the digital thermometer resolution correction $u(\Delta t_{res})$

The half limit due to the resolution of the digital thermometer is 0.05 °C. Assuming a rectangular distribution, the standard uncertainty of the digital thermometer resolution correction $u(\Delta t_{res})$ is :

$$u(\Delta t_{res}) = \frac{0.05}{\sqrt{3}} = 0.029\ °C$$

Uncertainty Budget or Summary Table

Source of Uncertainty	Symbol $U(x_i)$	Standard Uncertainty $u(x_i)$	Probability Distribution	Sensitivity Coefficient C_i	Uncertainty Contribution $u_i(y)=\lvert C_i\rvert \cdot u(x_i)$
Repeatability of thermocouple reading	$u(t_r)$	*0.033 °C*	Normal	*1*	*0.033 °C*
Thermocouple correction	$u(\Delta t_{tC})$	*0.5 °C*	Normal	*1*	*0.5 °C*
Thermocouple immersion correction	$u(\Delta t_{im})$	*0.058 °C*	Rectangular	*1*	*0.058 °C*
Thermocouple drift correction	$u(\Delta t_{der})$	*0.115 °C*	Rectangular	*1*	*0.115 °C*
Digital thermometer deviation correction	Δt_{ind}	*0.346 °C*	Rectangular	*1*	*0.346 °C*
Digital thermometer resolution correction	$u(\Delta t_{res})$	*0.029 °C*	Rectangular	*1*	*0.029 °C*
Result=	*400.5 °C*				***u(y)=0.623 °C***

The combined standards uncertainty $u_c(t_x)$ is:

$$u_c(t_x) = \sqrt{0.033^2 + 0.5^2 + 0.058^2 + 0.115^2 + 0.346^2 + 0.029^2}$$

$$u_c(t_x) = 0.623\ °C$$

<u>The expanded uncertainty:</u>

$U = kxu_c(t_x)$=2 x 0.623 °C = 1.3 °C

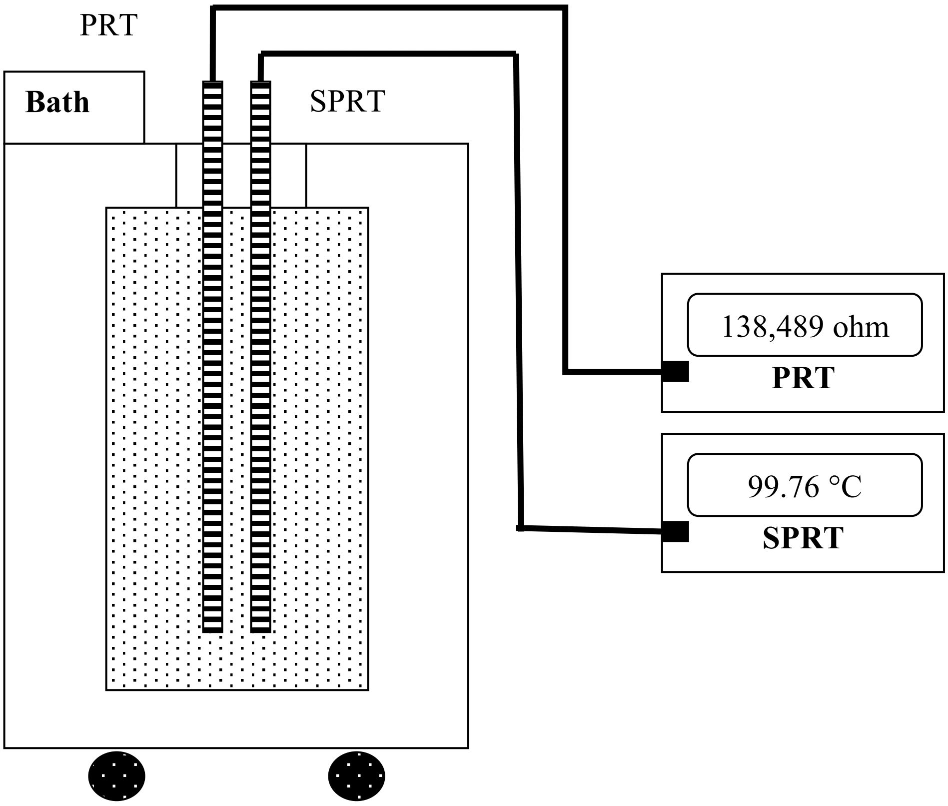
<u>The measurement result:</u>

The temperature of the chamber after taking into account the thermocouple correction is 400.5 ° C. The extended calibration uncertainty is ± 1.3 ° C, estimated at a confidence level of about 95% with a coverage factor k equal to 2.
The result of the temperature measurement of the test chamber is: 400.5 ° C ± 1.3 °C. The expanded uncertainty is indicated as the standard uncertainty of measurement multiplied by the coverage factor k = 2, which for a normal distribution corresponds to a probability of about 95%.

EXAMPLE 5: CALIBRATION OF A PLATINUM RESISTANCE PROBE

A 100-ohm semi-standard platinum resistance thermometer (PRT) was calibrated against a standard 25-ohm resistance thermometer (SPRT) traceable to the 1990 International Temperature Scale (ITS-90).

The schematic diagram of the Pt100 probe calibration is shown below.



Determination of the temperature of the calibration item

In all examples in this chapter, the calibration of the calibration item is carried out by the comparison method at a nominal temperature of 180 °C. The measurements are performed in a stirred oil thermostat without compensation block. As standard thermometer an SPRT (25 Ω) is used which was calibrated at the internal laboratory against two reference standards calibrated by the PTB. The resistance of the SPRT is determined with a resistance measuring bridge with direct temperature indication and a 100 Ω standard resistor, which were both calibrated by a DKD laboratory.

The temperature at which the calibration item is calibrated is determined by measurement with the standard thermometer and by additional corrections:

$$t_x = t_N + \delta t_{Kal} + \delta t_{Drift} + c_R\, \delta R_R + \delta t_{Br} + \delta t_{WaN} + \delta t_{EWN} + \delta t_{WAP} + \delta t_{Hom} + \delta t_{Stab}$$

with

t_x	temperature of the calibration item according to ITS-90
t_N	mean value of the temperature of the SPRT
δt_{Kal}	correction due to the measurement uncertainty in the calibration of the SPRT
δt_{Drift}	correction due to a possible drift of the SPRT since the last calibration
δR_R	correction due to the measurement uncertainty in the calibration of the standard resistor
δt_{Br}	correction due to the measurement uncertainty in the calibration of the resistance measuring bridge
δt_{WaN}	correction due to a possible heat conduction by the SPRT

δt_{EWN} correction for the self-heating of the SPRT

δt_{WAP} correction due to a possible heat conduction by the calibration item

δt_{Hom} correction due to inhomogeneities in the thermostat

δt_{Stab} correction due to temporal instabilities in the bath

c_R sensitivity of the measuring bridge; in the range selected, $c_R \cong 10$ K/∧ is valid. The corrections given in this list are in most cases not known and presumably very small. As best estimate a correction of 0 K is usually assumed which is, however, affected by an uncertainty. In detail, the contributions were determined as follows:

t_N mean value of the temperature of the standard thermometer (SPRT): The measuring bridge calculates the temperature from the entered coefficients of the deviation function which were determined in the calibration, and calculates the mean value of ten individual measurements and the standard deviation of this mean value. As result of the measurement, a mean temperature of 180.234 °C is indicated, with a standard deviation of the mean value of 1.2 mK.

δt_{Kal} correction due to the measurement uncertainty in the calibration of the SPRT: According to the calibration certificate, the measurement uncertainty of the SPRT at 180 °C is 15 mK (k = 2), so the standard uncertainty is 7.5 mK.

δt_{Drift} correction due to a possible drift of the thermometer since the last calibration: From the known history of the thermometer it is concluded that the drift since the last calibration will not be greater than ±6 mK. From this a standard uncertainty of 6 mK / $\sqrt{3}$ = 3.5 mK follows.

δR_R correction due to the measurement uncertainty in the calibration of the standard resistor: The relative measurement uncertainty of the standard resistor is given in the calibration certificate as 3.10^{-6} (k = 2). For an actual resistance of the SPRT of approx. 43 Ω, this corresponds to an uncertainty of
0.13 mΩ (k = 2) and to a standard uncertainty of 0.07 mΩ. Experience has shown that the drift of the resistor since the last calibration can be neglected.

δt_{Br} correction due to the measurement uncertainty of the resistance measuring bridge. For the measurement range used, the calibration certificate states an expanded uncertainty (k = 2) of 3 mK. The indication of the bridge shows six digits but at the interface to the data acquisition seven digits are available over which temporal averaging is carried out. So measurement uncertainties due to the limited resolution can be neglected in contrast to the other contributions to the measurement uncertainty.

δt_{WaN} correction due to a possible heat conduction by the SPRT: Pulling the SPRT 20 mm out of the bath led to a temperature change of 2 mK (which due to the temperature variations of the bath could be estimated only inaccurately). From this a standard uncertainty of 2 mK / $\sqrt{3}$ = 1.2 mK follows.

δt_{EWN} correction for self-heating of the SPRT: The calibration certificate states that a measurement current of 1 mA in a water triple point cell has led to a heating of 2.1 mK. This contribution is neglected in the following as the thermometer is both calibrated and used now at a measurement current of 1 mA.

δt_{WAP} correction due to a possible heat conduction by the calibration item: Pulling the calibration item 20 mm out of the bath led to a temperature change of 1 mK (which due to the temperature variations of the bath could be estimated only inaccurately), measured with the resistance bridge. This contribution is neglected. In part of the examples, it would not have been possible to detect any effect due to the low resolution of the calibration items.

t_{Hom} correction due to inhomogeneities in the thermostat: It is known from previous investigations that the temperature difference between calibration item and standard thermometers due to inhomogeneities in the bath can amount to ±8 mK at most. From this a standard uncertainty of 8 mK / $\sqrt{3}$ = 4.6 mK follows.

δt_{Stab} correction due to temporal instabilities in the bath: It is known from previous investigations that the temperature difference between calibration item and standard thermometers due to temporal instabilities in the bath can amount to ±6 mK at most. From this a standard uncertainty of 6 mK / $\sqrt{3}$ = 3.5 mK follows.

The individual contributions to the uncertainty of the temperature of the calibration item are summarised in Table below.

Quantity	Brief description	Estimate	Standard uncertainty	Distribution	Sensitivity coefficient	Uncertainty contribution
t_N	Dispersion of measurement values - SPRT	180.234 °C	1.2 mK	normal	1	1.2 mK
δt_{Kal}	Calibration - SPRT	0 K	7.5 mK	normal	1	7.5 mK
δt_{Drift}	Drift - standard thermometer	0 K	3.5 mK	rectan-gular	1	3.5 mK
δR_R	Standard resistor	0 Ω	0.07 mΩ	normal	10 K/Ω	0.7 mK
δt_{Br}	Measuring bridge	0 K	1.5 mK	normal	1	1.5 mK
δt_{WaN}	Heat dissipation - SPRT	0 K	1.2 mK	rectan-gular	1	1.2 mK
δt_{Hom}	Homogeneity - thermostat	0 K	4.6 mK	rectangular	1	4.6 mK
δt_{Stab}	Stability - thermostat	0 K	3.5 mK	rectangular	1	3.5 mK
t_x	Temperature - calibration item	180.234 °C	10.3 mK			

Table: Uncertainty of the temperature of the calibration item

Calibration of a precision resistance thermometer with an ohmmeter

At the temperature t_x, the resistance of the calibration item (Pt100 precision thermometer) is measured. The measurement of the resistance of the calibration item is made with five digits calibrated resistance measuring instrument (ohmmeter) for which a Calibration Certificate is available. The model for this measurement is obtained as follows:

$$R(t_x) = R_W + \delta R_{Ohm} + \delta R_{Drift} + \delta R_{Auf} + \delta R_{Par} + c_t \cdot \delta T + \delta R_{Hys}$$

with

R_W indication of the ohmmeter

δR_{Ohm} correction due to the measurement uncertainty in the calibration of the ohmmeter

δR_{Drift} correction due to the drift of the ohmmeter since the last calibration

δR_{Auf} correction due to the limited resolution of the ohmmeter

δR_{Par} correction due to parasitic thermal voltages

δT correction due to the uncertainty of the temperature of the calibration item

c_t sensitivity of the thermometer, here: 0.4 Ω/K

δR_{Hys} correction due to hysteresis effects

These contributions were in detail determined as follows:

R_W indication of the ohmmeter: The ohmmeter displays a value of 168.43 Ω. The standard deviation of the mean value from several measurements is determined to be 0.005 Ω.

δR_{Ohm} correction due to the measurement uncertainty in the calibration of the ohmmeter: According to the calibration certificate, the measurement uncertainty of the ohmmeter is 0.020 Ω (k = 2) and the standard uncertainty thus is 10 mΩ.

δR_{Drift} correction due to the drift of the ohmmeter since the last calibration: Due to the known history of the ohmmeter it is ensured that the drift since the last calibration is ±20 mΩ at most. From this a standard uncertainty of 20 mΩ/√3 = 11.5 mΩ follows.

δR_{Auf} correction due to the limited resolution of the ohmmeter: The limited resolution of the ohmmeter of 0.01 Ω allows a reading within ±0.005 Ω. From this a standard uncertainty of 5 mΩ / √3 = 2.9 mΩ follows.

δR_{Par} correction due to parasitic thermal voltages. The influence of parasitic thermal voltages was determined by reversal on the ohmmeter. Due to the limited resolution of the ohmmeter, an effect could not be detected and can therefore be neglected.

δT correction due to the uncertainty of the temperature of the calibration item: In Table A.1 the uncertainty of the temperature of the calibration item was determined at 10.3 mK.

δR_{Hys} correction due to hysteresis effects: Two measurements were carried out. For one measurement the thermometer had previously been in a salt bath at 250 °C and for the other measurement at 0 °C. The results differed by 22 mΩ. From this a contribution to the measurement uncertainty of 22 mΩ / 2√3 = 6.4 mΩ follows.

These contributions are summarised in Table below: **The measurement uncertainty of the platinum resistance thermometer to be calibrated**

Quantity	**Brief description**	**Estimate**	**Standard uncertainty**	**Distribution**	**Sensitivity coefficient**	**Uncertainty contribution**
R_W	Reading - ohmmeter	168.43 Ω	5 mΩ	normal	1	5 mΩ
δR_{Ohm}	Calibration-ohmmeter	0 Ω	10 mΩ	normal	1	10.0 mΩ
δR_{Drift}	Drift - ohmmeter	0 Ω	11.5 mΩ	rectan-gular	1	11.5 mΩ
δR_{Auf}	Resolution - ohmmeter	0 Ω	2.9 mΩ	rectan-gular	1	2.9 mΩ
δR_{Hys}	Hysteresis effects	0 Ω	6.4 mΩ	rectan-gular	1	6.4 mΩ
δT	Temperature - calibration item	0 K	10.3 mK	normal	0.4 Ω/K	4.1 mΩ
$R(t_x)$		168.43 Ω	18.0 mΩ			
$R(t_x)$					k = 2	33.6 mΩ

Measurement Result: The resistance of the IPRT at the temperature of 180.234 °C is 168.43 Ω. The measurement uncertainty is 0.04 Ω. This corresponds to an uncertainty of the temperature measurement of 0.09 °C.

The uncertainty stated is the expanded uncertainty which is obtained from the standard uncertainty by multiplication by the coverage factor k = 2. The value of the measurand lies with a probability of 95 % within the interval of values assigned.

EXAMPLE 6: CALIBRATION OF A WEIGHT OF NOMINAL VALUE 10 KG

The calibration of a nominal weight of 10 kg of class OIML M1 is carried out by comparison with a reference standard (class OIML F2) of the same nominal value using a mass comparator whose characteristics of performance have already been determined.

The unknown conventional mass m_x is obtained from:

$$m_x = m_S + \delta m_D + \delta m + \delta m + \delta m_C + \delta B$$

Where:

m_S conventional mass of the reference standard,
δm_D drift from the value of the standard since its last calibration,
δm observed difference in mass between the unknown mass and the reference standard,
δm_C correction due to eccentricity and magnetic effects,
δB correction of the air buoyancy.

Reference standard (m_S): The calibration certificate for the reference standard gives a value of 10,000.005 g with an associated expanded uncertainty of 45 mg (with a coverage factor of k = 2).

The deviation of the value of the reference standard (m_D): The drift of the value of the reference standard is estimated from previous calibrations estimated at the limits of **± 15 mg.**

The comparator (δm,δm_C): resolution of the comparator = 0.005 g, A preliminary evaluation of the repeatability of the mass difference between two weights of the same nominal value gives a common estimate of the standard deviation of 25 mg. No correction is applied for the comparator, while variations due to eccentricity and magnetic effects are estimated at limits ± 10 mg with a rectangular distribution.

The air buoyancy (δB): No correction is made for the effect of the air buoyancy; the gap limits are estimated at ± 1x10^{-6} of the nominal value.

Correlation: None of the input quantities is considered significantly correlated.

Measurements made: Three observations of the difference in mass between the unknown mass and the reference standard are obtained by using the substitution method and the ABBA ABBA ABBA substitution scheme:

N°	Conventional mass		Reading		Measured difference
1	Reference standard (E_1)		+0.010g		+0.01g
	Mass (M_1)		+0.020g		
	Mass (M_2)		+0.025g		
	Reference standard (E_2)		+0.015g		
2	Reference standard (E_1)		+0.025g		+0.03g

	Mass (M_1)		+0.050g		
	Mass (M_2)		+0.055g		
	Reference standard (E_2)		+0.020g		
3	Reference standard (E_1)		+0.025g		+0.02g
	Mass (M_1)		+0.045g		
	Mass (M_2)		+0.040g		
	Reference standard (E_2)		+0.020g		

Arithmetic average: $\overline{\delta m}$ = 0.020 g

Estimated standard deviation (Obtained from previous evaluation):

$$S_p(\delta m) = 25mg$$

Standard uncertainty of repeatability:

$$u(\delta m) = u(\overline{\delta m}) = \frac{25mg}{\sqrt{3}} = 14.4\,mg$$

Budget or uncertainty budget

Input quantity X_i	Estimation x_i	Standard Uncertainty $u(x_i)$	Probability Distribution	Sensitivity Coefficient C_i	Uncertainty Contribution $u_i(y)=\lvert C_i \rvert \cdot u(x_i)$
m_S	*10 000.005 g*	*22.5 mg*		*1*	*22.5 mg*
δm_D	*0.000 g*	*8.66 mg*		*1*	*8.66 mg*
δm	*0.020 g*	*14.4 mg*		*1*	*14.4 mg*
δm_C	*0.000 g*	*5.77 mg*		*1*	*5.77 mg*
δB	*0.000 g*	*5.77 mg*		*1*	*5.77 mg*
m_x	*10 000.025 g*				*u(y)=29.3 mg*

Expanded uncertainty:

U=kxu(m_x)=2x29.3mg= 59 mg

Measurement result:

The measurement result of the mass of nominal mass 10 kg is: 10 000 025 kg ± 59 mg.
The expanded uncertainty is indicated as the standard uncertainty of measurement multiplied by the coverage factor k = 2, which for a normal distribution corresponds to a probability of about 95%.

EXAMPLE 7: MEASURING THE RESISTANCE AND REACTANCE OF A CIRCUIT (CASES OF COROLATED VARIABLES)

The resistance R and the reactance X of a circuit element are determined by measuring the amplitude V of a sinusoidally-alternating potential difference across its terminals, the amplitude I of the alternating current passing through it, and the phase-shift angle φ of the alternating potential difference relative to the alternating current.

Thus, the three input quantities are V, I, and φ and the three output quantities: the measurands are the three impedance components R, X, and Z. Since $Z^2 = R^2 + X^2$, there are only two independent output quantities.

The measurands are related to the input quantities by Ohm's law:

$$R = \frac{V}{I} cos\varphi$$

$$X = \frac{V}{I} sin\varphi$$

$$Z = \frac{V}{I}$$

$$s(\bar{q}, \bar{r}) = \frac{1}{n(n-1)} \sum_{k=1}^{n} (q_k - \bar{q})(r_k - \bar{r})$$

And

$$r(x_i, x_j) = r(\bar{X}_i, \bar{X}_j) = s(\bar{X}_i, \bar{X}_j)/[s(\bar{X}_i)s(\bar{X}_j)]$$

Values of the input quantities V, I, and φ obtained from five sets of simultaneous observations

Set number k	**Input quantities**		
	V (V)	I (mA)	ϕ (rad)
1	5.007	19.663	1.0456
2	4.994	19.639	1.0438
3	5.005	19.640	1.0468
4	4.990	19.685	1.0428
5	4.999	19.678	1.0433
Arithmetic mean	$\bar{V}$= 4.999 0	$\bar{I}$= 19.661 0	φ= 1.044 46
Experimental standard deviation of mean	$s(\bar{V})$= 0.003 2	$s(\bar{I})$= 0.009 5 5	$s(\bar{\varphi})$= 0.000 75
Correlation coefficients			
$r(V, I) = -0.36$			
$r(V, \varphi) = 0.86$			
$r(I, \varphi) = -0.65$			

Value of output quantities R, X **et** Z **:**

Set number Input quantities k	**Output quantities**		
	$R = (V/I)$ cos ϕ (Ω)	$X = (V/I)$ sin ϕ (Ω)	$Z = V/I$ (Ω)
1	127.67	220.32	254.64
2	127.89	219.79	254.29
3	127.51	220.64	254.84
4	127.71	218.97	253.49
5	127.88	219.51	254.04
Arithmetic mean	R = 127.732	X = 219.847	Z = 254.260

Experimental standard deviation of mean	$s(R) = 0.071$	$s(X) = 0.295$	$s(Z) = 0.236$
Correlation coefficients			
$r(R, X) = -0.588$			
$r(R, Z) = -0.485$			
$r(X, Z) = 0.993$			

The values of the three measurands R, X and Z are obtained from the relations given in the equation using the average values $\bar{V}$, $\bar{I}$ et $\bar{\varphi}$.

$R = \frac{\bar{V}}{\bar{I}} cos\bar{\varphi}$	$X = \frac{\bar{V}}{\bar{I}} sin\bar{\varphi}$	$Z = \frac{\bar{V}}{\bar{I}}$

When applying the law of propagation of uncertainties for correlated variables:

$$u_C^2(Z) = \left(\frac{1}{\bar{I}}\right)^2 .u^2(\bar{V}) + \left(\frac{\bar{V}}{\bar{I}^2}\right)^2 .u^2(\bar{I}) + 2\left(\frac{1}{\bar{I}}\right)\left(-\frac{\bar{V}}{\bar{I}^2}\right)u(\bar{V})u(\bar{I})r(\bar{V},\bar{I})$$

$$u_C^2(Z) = Z^2\left[\frac{u(\bar{V})}{\bar{V}}\right]^2 + Z^2\left[\frac{u(\bar{I})}{\bar{I}}\right]^2 - 2Z^2\left[\frac{u(\bar{V})}{\bar{V}}\right]\left[\frac{u(\bar{I})}{I}\right]r(\bar{V},\bar{I})$$

$$u_{C,r}^2(\bar{Z}) = u_r^2(\bar{V}) + u_r^2(\bar{I}) - 2u_r(\bar{V})u_r(\bar{I})r(\bar{V},\bar{I})$$

$$u_{C,r}^2(\bar{Z}) = 0.236\ \Omega$$

Calculated values of the output quantities R, X, and Z: approach

Measurand Index l	**Relationship between estimate of measurand y_l and input estimates x_i**	**Value of estimate y_l which is the result of measurement**	**Combined standard Uncertainty $u_c(y_l)$ of result of measurement**
1	$R = (V/I)$ cos ϕ	$R = 127.732\ \Omega$	$u_c(R) = 0.071\ \Omega$ $u_c(R)/R = 0.06 \times 10^{-2}$
2	$X = (V/I)$ sin ϕ	$X = 219.847\ \Omega$	$u_c(X) = 0.295\ \Omega$ $u_c(X)/X = 0.13 \times 10^{-2}$
3	$Z = V/I$	$Z = 254.260\ \Omega$	$u_c(Z) = 0.236\ \Omega$ $u_c(Z)/Z = 0.09 \times 10^{-2}$

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